DISTRIBUTIONAL STABILITY OF THE SZAREK AND BALL INEQUALITIES

ALEXANDROS ESKENAZIS, PIOTR NAYAR, AND TOMASZ TKOCZ

Abstract. We prove an extension of Szarek's optimal Khinchin inequality (1976) for distributions close to the Rademacher one, when all the weights are uniformly bounded by a $1/\sqrt{2}$ fraction of their total ℓ_2-mass. We also show a similar extension of the probabilistic formulation of Ball's cube slicing inequality (1986). These results establish the distributional stability of these optimal Khinchin-type inequalities. The underpinning to such estimates is the Fourier-analytic approach going back to Haagerup (1981).

2010 Mathematics Subject Classification. Primary 60E15; Secondary 42A38, 26D15, 60G50.

Key words. Khinchin inequality, sums of independent random variables, moment comparison, cube slicing.

1. Introduction

Let $\varepsilon_1, \varepsilon_2, \ldots$ be independent identically distributed (i.i.d.) Rademacher random variables, that is, symmetric random signs satisfying $\mathbb{P}(\varepsilon_j = \pm 1) = \frac{1}{2}$. Motivated by his study of bilinear forms on infinitely many variables, Littlewood conjectured in [26] (see also [15]) the following inequality: for every $n \geq 1$ and every unit vector a in $\mathbb{R}^n$, we have

$$\mathbb{E}\left|\sum_{j=1}^n a_j\varepsilon_j\right| \geq \mathbb{E}\left|\frac{\varepsilon_1+\varepsilon_2}{\sqrt{2}}\right| = \frac{1}{\sqrt{2}}, \tag{1}$$

which is clearly best possible. Not until 46 years after it had been posed, was this proved by Szarek in [34]. His result was later generalised in a stunning way to the setting of vector-valued coefficients a_j in arbitrary normed space by Latała and Oleszkiewicz in [24] (see also [30, Section 4.2] for a modern presentation of their proof using discrete Fourier analysis). Szarek's original proof was based mainly on an intricate inductive scheme (see also [35]). Note that (1) holds trivially if $\|a\|_\infty = \max_j |a_j| \geq \frac{1}{\sqrt{2}}$, for if, say we have $|a_1| \geq \frac{1}{\sqrt{2}}$, then thanks to independence and convexity,

$$\mathbb{E}\left|\sum_{j=1}^n a_j\varepsilon_j\right| \geq \mathbb{E}\left|a_1\varepsilon_1 + \mathbb{E}\sum_{j=2}^n a_j\varepsilon_j\right| = \mathbb{E}|a_1\varepsilon_1| = |a_1| \geq \frac{1}{\sqrt{2}}.$$

Haagerup in his pioneering work [14] on Khinchin inequalities offered a very different approach to the nontrivial regime $\|a\|_\infty \leq \frac{1}{\sqrt{2}}$, using classical Fourier-analytic integral representations along with tricky estimates for a special function.

This material is based upon work supported by the NSF grant DMS-1929284 while A. E. was in residence at ICERM for the Harmonic Analysis and Convexity program. P.N.'s research was supported by the National Science Centre, Poland, grant 2018/31/D/ST1/0135. T.T.'s research was supported by the NSF grant DMS-1955175.

Taking that route, the point of this paper is to illustrate the robustness of Haagerup's method and extend (1) to i.i.d. sequences of random variables whose distribution is *close* to the Rademacher one in the W_2-Wasserstein distance. Using the same framework, we also treat Ball's cube slicing inequality from [2] which asserts that the maximal-volume hyperplane section of the cube $[-1,1]^n$ in $\mathbb{R}^n$ is attained at $(1,1,0,\dots,0)^\perp$. This can be equivalently stated in probabilistic terms as an inequality akin to (1) as follows (see, e.g. equation (2) in [6]). Let $\xi_1, \xi_2, \dots$ be i.i.d. random vectors uniform on the unit Euclidean sphere in $\mathbb{R}^3$. For every $n \geq 1$ and every unit vector a in $\mathbb{R}^n$, we have

$$\mathbb{E}\left[\left|\sum_{j=1}^n a_j\xi_j\right|^{-1}\right] \leq \mathbb{E}\left[\left|\frac{\xi_1+\xi_2}{\sqrt{2}}\right|^{-1}\right] = \sqrt{2}, \tag{2}$$

where here and throughout $|\cdot|$ denotes the standard Euclidean norm.

Szarek's inequality (1), Balls inequality (2), as well as these extensions fall under the umbrella of so-called Khinchin-type inequalities. The archetype was Khinchin's result asserting that all L_p norms of Rademacher sums $\sum a_j\varepsilon_j$ are comparable to its L_2-norm, established in his work [22] on the law of the iterated logarithm (and perhaps discovered independently by Littlewood in [26]). Due to the intricacies of the methods involved, sharp Khinchin inequalities are known only for a handful of distributions, most notably random signs ([14, 29]), but also uniforms ([4, 5, 6, 8, 18, 21, 25]), type L ([17, 32]), Gaussian mixtures ([1, 10]), marginals of ℓ_p-balls ([3, 11]), or distributions with good spectral properties ([23, 33]). The present work makes a first step towards more general distributions satisfying only a closeness-type assumption instead of imposing structural properties. Viewing sharp Khinchin-type inequalities as maximization problems for functionals on the sphere, our results assert, perhaps surprisingly, the fact that such inequalities are stable with respect to perturbations of the law of the underlying random vectors. These *distributional stability* results are novel in the context of optimal probabilistic inequalities.

2. MAIN RESULTS

For $p > 0$ and a random vector X in $\mathbb{R}^d$, we denote its L_p-norm with respect to the standard Euclidean norm $|\cdot|$ on $\mathbb{R}^d$ by $\|X\|_p = (\mathbb{E}|X|^p)^{1/p}$, whereas for a (deterministic) vector a in $\mathbb{R}^n$, $\|a\|_\infty = \max_{j\leq n}|a_j|$ is its ℓ_∞-norm. We say that the random vector X in $\mathbb{R}^d$ is symmetric if $-X$ has the same distribution as X. We also recall that the vector X is called rotationally invariant if for every orthogonal map U on $\mathbb{R}^d$, UX has the same distribution as X. Equivalently, X has the same distribution as $|X|\xi$, where ξ is uniformly distributed on the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$ and independent of $|X|$. Recall that the W_2-Wasserstein distance $\mathsf{W}_2(X,Y)$ between (the distributions of) two random vectors X and Y in $\mathbb{R}^d$ is defined as $\inf_{(X',Y')}\|X'-Y'\|_2$, where the infimum is taken over all couplings of X and Y, that is, all random vectors (X',Y') in $\mathbb{R}^{2d}$ such that X' has the same distribution as X and Y' has the same distribution as Y.

Our first result is an extension of Szarek's inequality (1) which reads as follows.

Theorem 1. *There is a positive universal constant δ_0 such that if we let $X_1, X_2, \dots$ be i.i.d. symmetric random variables satisfying*

$$\big\| |X_1| - 1 \big\|_2 \leq \delta_0, \tag{3}$$

then for every $n \geq 3$ and unit vectors a in $\mathbb{R}^n$ with $\|a\|_\infty \leq \frac{1}{\sqrt{2}}$, we have

$$\mathbb{E}\left|\sum_{j=1}^n a_j X_j\right| \geq \mathbb{E}\left|\frac{X_1 + X_2}{\sqrt{2}}\right|. \tag{4}$$

Moreover, we can take $\delta_0 = 10^{-4}$.

Note that left hand side of (3) is nothing but the W_2-Wasserstein distance between the distribution of X_1 and the Rademacher distribution since $|x \pm 1| \geq \big||x| - 1\big|$ for $x \in \mathbb{R}$ and thus the optimal coupling of the two distributions is $\big(X_1, \mathrm{sign}(X_1)\big)$.

Our second main result provides an analogous extension for Ball's inequality (2).

Theorem 2. *Let $X_1, X_2, \dots$ be i.i.d. symmetric random vectors in $\mathbb{R}^3$. Suppose their common characteristic function $\phi(t) = \mathbb{E}e^{i\langle t, X_1\rangle}$ satisfies*

$$|\phi(t)| \leq \frac{C_0}{|t|}, \qquad t \in \mathbb{R}^3 \setminus \{0\}, \tag{5}$$

for some constant $C_0 > 0$. Assume that

$$\mathsf{W}_2(X_1, \xi) \leq 10^{-38} C_1^{-9} \min\left\{(\mathbb{E}|X_1|^3)^{-6}, 1\right\}, \tag{6}$$

where $C_1 = \max\{C_0, 1\}$ and ξ is a random vector uniform on the unit Euclidean sphere $\mathbb{S}^2$ in $\mathbb{R}^3$. Then for every $n \geq 3$ and unit vectors a in $\mathbb{R}^n$ with $\|a\|_\infty \leq \frac{1}{\sqrt{2}}$, we have

$$\mathbb{E}\left|\sum_{j=1}^n a_j X_j\right|^{-1} \leq \mathbb{E}\left|\frac{X_1 + X_2}{\sqrt{2}}\right|^{-1}. \tag{7}$$

Plainly, if we know that X_1 and ξ are sufficently close in W_3, then the parameter $\mathbb{E}|X_1|^3$ in (6) is redundant. In contrast to Theorem 1, here the closeness assumption (6) is put in terms of two parameters of the distribution: its third moment and the polynomial decay of its characteristic function. It is not clear whether this is essential. At the technical level of our proofs, the third moment is needed to carry out a certain Gaussian approximation, whilst the decay assumption has to do with an a priori lack of integrability in the Fourier-analytic representation of the L_{-1} norm (as opposed to the L_1-norm handled in Theorem 1).

On the other hand, neither of these is very restrictive. In particular, if X_1 has a density f on $\mathbb{R}^3$ vanishing at ∞ whose gradient is integrable, then

$$\begin{aligned} |t||\phi(t)| \leq \sum_{j=1}^3 |t_j\phi(t)| = \sum_{j=1}^3 \left|\int_{\mathbb{R}^3} t_j e^{i\langle t,x\rangle} f(x)\mathrm{d}x\right| &= \sum_{j=1}^3 \left|\int_{\mathbb{R}^3} i e^{i\langle t,x\rangle}\partial_j f(x)\mathrm{d}x\right| \\ &\leq \sqrt{3}\int_{\mathbb{R}^3} |\nabla f(x)|\mathrm{d}x, \end{aligned}$$

so (5) holds with $C_0 = \sqrt{3}\int_{\mathbb{R}^3}|\nabla f|$.

Another natural sufficient condition is the rotational invariance of X_1: if, say, X_1 has the same distribution as $R\xi$, for a nonnegative random variable R and an independent of it random vector ξ uniform on the unit sphere $\mathbb{S}^2$, then Archimedes' Hat-Box theorem implies that $\langle t, R\xi\rangle$, conditioned on the value of R, is uniform on $[-R|t|, R|t|]$ and thus

$$|\phi(t)| = |\mathbb{E}_R\mathbb{E}_\xi e^{i\langle t,R\xi\rangle}| = \left|\mathbb{E}_R\frac{\sin(R|t|)}{R|t|}\right| \leq \frac{\mathbb{E}R^{-1}}{|t|} = \frac{\mathbb{E}|X_1|^{-1}}{|t|}.$$

Moreover, in this case $\mathsf{W}_2(X_1,\xi) = \|R-1\|_2$ (since for every unit vectors θ, θ' in $\mathbb{R}^d$ and $R \geq 0$, we have $|R\theta - \theta'| \geq |R-1|$, as is easily seen by squaring). Probabilistically, this is an important special case as it yields results for symmetric unimodal distributions on $\mathbb{R}$. Indeed, if X is of the form $R\xi$ as above, for $q > -1$, we have the identity

$$\mathbb{E}\left|\sum_{j=1}^n a_jX_j\right|^q = \mathbb{E}\left|\sum_{j=1}^n a_jR_j\xi_j\right|^q = (1+q)\mathbb{E}\left|\sum_{j=1}^n a_jR_jU_j\right|^q, \tag{8}$$

where the R_j are i.i.d. copies of R and the U_j are i.i.d. uniform random variables on $[-1,1]$, independent of the R_j (see Proposition 4 in [19]). The R_jU_j showing up in this formula can have any symmetric unimodal distribution, uniquely defined by the distribution of R_j. Thus, if $V_1, V_2, \dots$ be i.i.d. symmetric unimodal random variables, Theorem 2 then immediately yields a sharp upper bound on $\lim_{q\downarrow -1}(1+q)\mathbb{E}\left|\sum_{j=1}^n a_jX_j\right|^q$ for all unit vectors a with $\|a\|_\infty \leq \frac{1}{\sqrt{2}}$ (cf. [6, 5, 11, 25]).

A result in the same vein as Theorem 2 is König and Koldobsky's extension [19] of Ball's cube slicing inequality to product measures with densities satisfying certain regularity and moment assumptions. Their result also applies specifically to vectors of weights satisfying the small coefficient condition $\|a\|_\infty \leq \frac{1}{\sqrt{2}}$.

Approached differently, *full* extensions of (1) and (2) (i.e. without the small coefficient restriction on a) have been obtained in our recent work [12] for a very special family of distributions corresponding geometrically to extremal sections and projections of ℓ_p-balls.

3. PROOF OF THEOREM 1

Our approach builds on Haagerup's slick Fourier-analytic proof from [14]. We let

$$\phi(t) = \mathbb{E}e^{itX_1}, \qquad t \in \mathbb{R}, \tag{9}$$

be the characteristic function of X_1. Using the elementary Fourier-integral representation

$$|x| = \frac{1}{\pi}\int_{\mathbb{R}}(1-\cos(tx))t^{-2}\mathrm{d}t, \qquad x \in \mathbb{R},$$

as well as the symmetry and independence of the X_j, we have,

$$\mathbb{E}\left|\sum_{j=1}^n a_jX_j\right| = \frac{1}{\pi}\int_{\mathbb{R}}\Big(1 - \mathrm{Re}\ \mathbb{E}e^{it\sum a_jX_j}\Big)t^{-2}\mathrm{d}t = \frac{1}{\pi}\int_{\mathbb{R}}\left(1 - \prod_{j=1}^n\phi(a_jt)\right)t^{-2}\mathrm{d}t \tag{10}$$

(see also Lemma 1.2 in [14]). If a is a unit vector in $\mathbb{R}^n$ with nonzero components, using the AM-GM inequality, we obtain Haagerup's lower bound

$$\mathbb{E}\left|\sum_{j=1}^{n} a_j X_j\right| \geq \sum_{j=1}^{n} a_j^2 \Psi(a_j^{-2}), \tag{11}$$

where

$$\Psi(s) = \frac{1}{\pi}\int_{\mathbb{R}} \left(1 - \left|\phi\left(\frac{t}{\sqrt{s}}\right)\right|^s\right) t^{-2}\mathrm{d}t, \qquad s > 0. \tag{12}$$

(see Lemma 1.3 in [14]). The crucial lemma reads as follows.

Lemma 3. *Under the assumptions of Theorem 1, we have* $\Psi(s) \geq \Psi(2)$ *for every* $s \geq 2$.

If we take the lemma for granted, the proof of Theorem 1 is finished because the small coefficient assumption $\|a\|_\infty \leq \frac{1}{\sqrt{2}}$ gives $\Psi(a_j^{-2}) \geq \Psi(2)$ for each j, and as a result we get

$$\mathbb{E}\left|\sum_{j=1}^{n} a_j X_j\right| \geq \Psi(2) = \frac{1}{\pi}\int_{\mathbb{R}} \left(1 - \left|\phi\left(\frac{t}{\sqrt{2}}\right)\right|^2\right) t^{-2}\mathrm{d}t = \mathbb{E}\left|\frac{X_1 + X_2}{\sqrt{2}}\right|,$$

where the last equality is justified by (10).

It remains to prove Lemma 3. To this end, we recall that if the X_j were Rademacher random variables, then the special function Ψ becomes

$$\Psi_0(s) = \frac{1}{\pi}\int_{\mathbb{R}} \left(1 - \left|\cos\left(\frac{t}{\sqrt{s}}\right)\right|^s\right) t^{-2}\mathrm{d}t, \qquad s > 0. \tag{13}$$

Haagerup showed that for every $s > 0$,

$$\Psi_0(s) = \frac{2}{\sqrt{\pi s}}\frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} = \sqrt{\frac{2}{\pi}}\prod_{k=0}^{\infty}\left(1 - 1/(s+2k+1)^2\right)^{1/2} \tag{14}$$

and concluded by the product representation that Ψ_0 is strictly increasing. In particular, Lemma 3 holds in the Rademacher case due to monotonicity. The rest of the proof builds exactly on this observation: we show that the closeness of distributions guarantees that Ψ and Ψ_0 are close for, say $s \geq 3$, and that their derivatives are close for $2 \leq s \leq 3$. Crucially, not only do we know that Ψ_0 is strictly monotone, but also we can get a good bound on its derivative near the endpoint $s = 2$, which we record now for future use.

Lemma 4. *We have*

$$\inf_{2\leq s\leq 3} \Psi_0'(s) \geq \frac{\zeta(3) - 1}{8\sqrt{2}} = 0.01785... \tag{15}$$

Proof. Differentiating Haagerup's product expression (14) term-by-term yields

$$\begin{aligned}\Psi_0'(s) &= \frac{\mathrm{d}}{\mathrm{d}s}\sqrt{\frac{2}{\pi}}\prod_{k=0}^{\infty}\Big(1-(s+2k+1)^{-2}\Big)^{1/2} \\ &= \Psi_0(s)\sum_{k=0}^{\infty}\Big(1-(s+2k+1)^{-2}\Big)^{-1}(s+2k+1)^{-3} \\ &\geq \Psi_0(2)\sum_{k=0}^{\infty}(2k+4)^{-3} = \frac{1}{\sqrt{2}}\frac{\zeta(3)-1}{8}. \qquad \square\end{aligned}$$

The rest of this section is devoted to the proof of Lemma 3. We break it into several parts.

3.1. **A uniform bound on the characteristic function.**

Lemma 5. *Let X be a symmetric random variable satisfying* (3). *Then its characteristic function $\phi(t) = \mathbb{E}e^{itX}$ satisfies,*

$$|\phi(t)-\cos t| \leq \frac{\delta_0(\delta_0+2)}{2}t^2, \qquad t\in\mathbb{R}. \tag{16}$$

Proof. By symmetry, the triangle inequality and the bound $|\sin u| \leq |u|$, we get

$$\begin{aligned}|\phi(t)-\cos t| = |\mathbb{E}\left[\cos(t|X|)-\cos t\right]| &= 2\left|\mathbb{E}\left[\sin\left(t\frac{|X|-1}{2}\right)\sin\left(t\frac{|X|+1}{2}\right)\right]\right| \\ &\leq \frac{t^2}{2}\mathbb{E}\left[\big||X|-1\big|\cdot\big||X|+1\big|\right] \leq \frac{t^2}{2}\big\||X|-1\big\|_2\big\||X|+1\big\|_2,\end{aligned}$$

using the Cauchy-Schwarz inequality in the last estimate. Moreover,

$$\big\||X|+1\big\|_2 \leq \big\||X|-1\big\|_2 + 2.$$

Plugging in the assumption $\big\||X|-1\big\|_2 \leq \delta_0$ completes the proof. $\square$

3.2. **Uniform bounds on the special function and its derivative.**

Lemma 6. *Assuming* (3) *and the symmetry of X_1, the functions Ψ and Ψ_0 defined in* (12) *and* (13) *respectively satisfy*

$$|\Psi(s)-\Psi_0(s)| \leq \frac{2}{\pi}\sqrt{2\delta_0(\delta_0+2)}, \qquad s\geq 1. \tag{17}$$

Proof. Fix $T>0$. Breaking the integral defining Ψ into $\int_0^T + \int_T^\infty$ and using that $|a-b|\leq 1$ for $a,b\in[0,1]$, we obtain

$$\begin{aligned}|\Psi(s)-\Psi_0(s)| &= \frac{2}{\pi}\left|\int_0^\infty\left[\left|\phi\left(\frac{t}{\sqrt{s}}\right)\right|^s - \left|\cos\left(\frac{t}{\sqrt{s}}\right)\right|^s\right]t^{-2}\mathrm{d}t\right| \\ &\leq \frac{2}{\pi}\int_0^T\left|\left|\phi\left(\frac{t}{\sqrt{s}}\right)\right|^s - \left|\cos\left(\frac{t}{\sqrt{s}}\right)\right|^s\right|t^{-2}\mathrm{d}t + \frac{2}{\pi}\int_T^\infty t^{-2}\mathrm{d}t\end{aligned}$$

We also have $\big||a|^s-|b|^s\big| \leq s|a-b|$ for $a,b\in[-1,1]$, $s\geq 1$, thus Lemma 5 yields

$$|\Psi(s)-\Psi_0(s)| \leq \frac{2}{\pi}\int_0^T s\frac{\delta_0(\delta_0+2)}{2}\left(\frac{t}{\sqrt{s}}\right)^2 t^{-2}\mathrm{d}t + \frac{2}{\pi T} = \frac{2}{\pi}\left(T\frac{\delta_0(\delta_0+2)}{2}+\frac{1}{T}\right).$$

Optimizing over the parameter T gives the desired bound. $\square$

Lemma 7. *For $s \geq 2$ and $0 < u, v < 1$, we have*

$$|u^s \log u - v^s \log v| \leq |u - v|.$$

Proof. Let $f(x) = x^s \log x$. It suffices to prove that on $(0,1)$ we have $|f'(x)| \leq 1$, which is equivalent to $|\alpha t \log t + t| \leq 1$ with $t = x^{s-1} \in (0,1)$ and $\alpha = \frac{s}{s-1} \in [1,2]$. To prove this observe that for $t \in (0,1)$ we have $\alpha t \log t + t \leq t \leq 1$ and

$$\alpha t \log t + t \geq \alpha t \log t \geq -\frac{\alpha}{e} \geq -\frac{2}{e} > -1. \qquad \square$$

Lemma 8. *Assuming* (3) *and the symmetry of X_1, the functions Ψ and Ψ_0 defined in* (12) *and* (13) *satisfy*

$$|\Psi'(s) - \Psi_0'(s)| \leq 0.62\sqrt{\delta_0(\delta_0 + 2)}, \qquad s \geq 2. \tag{18}$$

Proof. Changing the variables and differentiating gives

$$\begin{aligned}\Psi'(s) &= \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{2}{\pi\sqrt{s}}\int_0^\infty \Big[1 - |\phi(t)|^s\Big] t^{-2}\mathrm{d}t\right) \\ &= -\frac{1}{2s}\Psi(s) - \frac{2}{\pi\sqrt{s}}\int_0^\infty |\phi(t)|^s \log|\phi(t)| t^{-2}\mathrm{d}t.\end{aligned}$$

Thus,

$$\begin{aligned}|\Psi'(s) - \Psi_0'(s)| \leq \frac{1}{2s}&|\Psi(s) - \Psi_0(s)| \\ &+ \frac{2}{\pi\sqrt{s}}\int_0^\infty \Big||\phi(t)|^s \log|\phi(t)| - |\cos(t)|^s \log|\cos(t)|\Big| t^{-2}\mathrm{d}t.\end{aligned}$$

To estimate the integral, we proceed along the same lines as in the proof of Lemma 6. We fix $T > 0$, write $\int_0^\infty = \int_0^T + \int_T^\infty$ and for the second integral use $|u^s \log u| = \frac{1}{s}|u^s \log(u^s)| \leq \frac{1}{es}$, $0 < u < 1$, to get a bound on it by $\frac{2}{esT}$, whilst for the first integral, using first Lemma 7 and then Lemma 5, we obtain

$$\begin{aligned}\int_0^T \Big||\phi(t)|^s \log|\phi(t)| - |\cos(t)|^s \log|\cos(t)|\Big| t^{-2}\mathrm{d}t &\leq \int_0^T |\phi(t) - \cos(t)| t^{-2}\mathrm{d}t \\ &\leq \frac{\delta_0(\delta_0 + 2)}{2}T,\end{aligned}$$

Altogether, with the aid of Lemma 6,

$$|\Psi'(s) - \Psi_0'(s)| \leq \frac{1}{2s}\frac{2}{\pi}\sqrt{2\delta_0(\delta_0 + 2)} + \frac{2}{\pi\sqrt{s}}\left(\frac{\delta_0(\delta_0 + 2)}{2}T + \frac{2}{esT}\right).$$

Minimising the second term over $T > 0$ leads to the bound by

$$\frac{1}{\pi s}\sqrt{2\delta_0(\delta_0 + 2)} + \frac{4}{\pi s}\sqrt{\frac{\delta_0(\delta_0 + 2)}{e}} = \frac{\sqrt{\delta_0(\delta_0 + 2)}}{\pi s}\left(\sqrt{2} + \frac{4}{\sqrt{e}}\right).$$

For $s \geq 2$, we have $\frac{1}{\pi s}\left(\sqrt{2} + \frac{4}{\sqrt{e}}\right) < 0.61...$ and this completes the proof. $\square$

3.3. **Proof of Lemma 3.** First we assume that $s \geq 3$. Using Lemma 6 and letting $\eta = \frac{2}{\pi}\sqrt{2\delta_0(\delta_0 + 2)}$ for brevity, we get

$$\Psi(s) \geq \Psi_0(s) - \eta.$$

Since Ψ_0 is increasing, $\Psi_0(s) \geq \Psi_0(3) = \Psi_0(3) - \Psi_0(2) + \Psi_0(2)$ and $\Psi_0(2) \geq \Psi(2) - \eta$, again using Lemma 6. Therefore,

$$\Psi(s) \geq \Psi(2) + \big(\Psi_0(3) - \Psi_0(2) - 2\eta\big).$$

It is now clear that as long as δ_0 is sufficiently small, namely $2\eta \leq \Psi_0(3) - \Psi_0(2)$, we get $\Psi(s) \geq \Psi(2)$, as desired. It can be checked that $\Psi_0(3) - \Psi_0(2) = \frac{4}{\pi\sqrt{3}} - \frac{1}{\sqrt{2}} = 0.027..$ and a choice of $\delta_0 \leq 10^{-4}$ suffices for the estimate $\Psi(s) \geq \Psi(2)$ to hold for $s \geq 3$.

Now we assume that $2 < s < 3$. We have

$$\Psi(s) = \Psi(2) + (s-2)\Psi'(\theta)$$

for some $2 < \theta < s$. Using Lemmas 8 and 4, we get

$$\Psi'(\theta) \geq \Psi_0'(\theta) - 0.62\sqrt{\delta_0(\delta_0+2)} \geq 0.017 - 0.62\sqrt{\delta_0(\delta_0+2)}$$

which is positive for all $\delta_0 \leq 3.7 \cdot 10^{-4}$. Thus, $\Psi(s) \geq \Psi(2)$ holds in both cases. □

4. PROOF OF THEOREM 2

The approach is the same as for Theorem 1, however certain technical details are substantially more involved. We begin with a Fourier-analytic representation for *negative* moments due to Gorin and Favorov [13].

Lemma 9 (Lemma 3 in [13]). *For a random vector X in $\mathbb{R}^d$ and $-d < q < 0$, we have*

$$\mathbb{E}|X|^q = \beta_{q,d} \int_{\mathbb{R}^d} \mathbb{E}e^{i\langle t,X\rangle} \cdot |t|^{-q-d} \mathrm{d}t, \tag{19}$$

where $\beta_{q,d} = 2^q \pi^{-d/2} \frac{\Gamma((d+q)/2)}{\Gamma(-q/2)}$, provided that the integral on the right hand side exists.

Specialised to $d = 3$, $q = -1$ ($\beta_{-1,3} = \frac{1}{2\pi^2}$) and $X = \sum_{j=1}^n a_j X_j$ with $X_1, \ldots, X_n$ independent random vectors, we obtain

$$\mathbb{E}\left|\sum_{j=1}^n a_j X_j\right|^{-1} = \frac{1}{2\pi^2}\int_{\mathbb{R}^3}\left(\prod_{j=1}^n \mathbb{E}e^{i\langle t, a_j X_j\rangle}\right)|t|^{-2}\mathrm{d}t. \tag{20}$$

Note that thanks to the decay assumption (5), the integral on the right hand side converges as long as $n \geq 2$ (assuming the a_j are nonzero). As in Ball's proof from [2], Hölder's inequality yields

$$\mathbb{E}\left|\sum_{j=1}^n a_j X_j\right|^{-1} \leq \prod_{j=1}^n \Phi\big(a_j^{-2}\big)^{a_j^2}, \tag{21}$$

where

$$\Phi(s) = \frac{1}{2\pi^2}\int_{\mathbb{R}^3}\left|\phi\left(s^{-1/2}t\right)\right|^s |t|^{-2}\mathrm{d}t, \qquad s > 1 \tag{22}$$

with

$$\phi(t) = \mathbb{E}e^{i\langle t, X_1\rangle}, \qquad t \in \mathbb{R}^3, \tag{23}$$

denoting the characteristic function of X_1. Exactly as in the proof of Theorem 1, the following pivotal lemma allows us to finish the proof.

Lemma 10. *Under the assumptions of Theorem 2, we have* $\Phi(s) \leq \Phi(2)$ *for every* $s \geq 2$.

If the X_j are uniform on the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$, we have $\phi(t) = \frac{\sin|t|}{|t|}$ (because $\langle t, X_1\rangle$ is uniform on $[-|t|, |t|]$), in which case the special function Φ defined in (22) becomes

$$\Phi_0(s) = \frac{2}{\pi}\int_0^\infty \left|\frac{\sin(s^{-1/2}t)}{s^{-1/2}t}\right|^s \mathrm{d}t, \qquad s > 1 \tag{24}$$

(after integrating in polar coordinates). Ball's celebrated integral inequality states that $\Phi_0(s) \leq \Phi_0(2)$, for all $s \geq 2$ (see Lemma 3 in [2], as well as [28, 31] for different proofs). Our proof of Lemma 10 relies on this, additional bounds on the derivative $\Phi_0'(s)$ near $s = 2$, as well as, crucially, bounds quantifying how close Φ is to Φ_0. In the following subsections we gather such results and then conclude with the proof of Lemma 10.

4.1. **A uniform bound on the characteristic function.**

Throughout these sections ξ always denotes a random vector uniform on the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$.

Lemma 11. *Let X be a symmetric random vector in $\mathbb{R}^3$ with $\delta = \mathsf{W}_2(X, \xi)$. Then, its characteristic function $\phi(t) = \mathbb{E}e^{i\langle t, X\rangle}$ satisfies*

$$\left|\phi(t) - \frac{\sin|t|}{|t|}\right| \leq \frac{\delta(\delta+2)}{2}|t|^2, \qquad t \in \mathbb{R}^3. \tag{25}$$

Proof. Let ξ be uniform on $\mathbb{S}^2$ such that for the joint distribution of (X, ξ), we have $\|X - \xi\|_2 = \mathsf{W}_2(X, \xi) = \delta$. By symmetry, the bound $|\sin u| \leq |u|$ and the Cauchy-Schwarz inequality (used twice), we get

$$\begin{aligned}\left|\phi(t) - \frac{\sin|t|}{|t|}\right| &= \left|\mathbb{E}\left[\cos\langle t, X\rangle - \cos\langle t, \xi\rangle\right]\right| \\ &= 2\left|\mathbb{E}\left[\sin\left(\tfrac{1}{2}\langle t, X - \xi\rangle\right)\sin\left(\tfrac{1}{2}\langle t, X + \xi\rangle\right)\right]\right| \\ &\leq \frac{|t|^2}{2}\mathbb{E}\left[|X - \xi| \cdot |X + \xi|\right] \\ &\leq \frac{|t|^2}{2}\|X - \xi\|_2\|X + \xi\|_2.\end{aligned}$$

To conclude we use the triangle inequality

$$\|X + \xi\|_2 \leq \|X - \xi\|_2 + 2\|\xi\|_2 = \|X - \xi\|_2 + 2. \qquad \square$$

4.2. **Bounds on the special function.**

We begin with a bound on the difference $\Phi(s) - \Phi_0(s)$ obtained from the uniform bound on the characteristic functions (Lemma 11 above). In contrast to Lemma 6, the bound is not uniform in s. For s *not* too large (the bulk), we incur the factor $s^{3/4}$. To fight it off for large values of s, we shall employ a Gaussian approximation. For that part to work, it is crucial that $\Phi_0(2) - \Phi_0(\infty) = \sqrt{2} - \sqrt{\frac{6}{\pi}} > 0$.

4.2.1. *The bulk.*

Lemma 12. *Let X be a symmetric random vector in $\mathbb{R}^3$ with $\delta = \mathsf{W}_2(X,\xi)$ and characteristic function ϕ satisfying* (5) *for some $C_0 > 0$. Let Φ and Φ_0 be defined through* (22) *and* (24) *respectively. For every $s \geq 2$, we have*

$$|\Phi(s) - \Phi_0(s)| \leq \frac{2^{11/4}}{3\pi} s^{3/4} \big(\delta(\delta+2)\big)^{1/4} \big(C_0^2+1\big)^{3/4}. \tag{26}$$

Proof. Given the definitions, we have

$$\Phi(s) - \Phi_0(s) = \frac{\sqrt{s}}{2\pi^2} \int_{\mathbb{R}^3} \left(|\phi(t)|^s - \left|\frac{\sin|t|}{|t|}\right|^s \right) |t|^{-2} \mathrm{d}t.$$

We fix $T > 0$ and split the integration into two regions.
Small t. Using Lemma 11 and $||a|^s - |b|^s| \leq s|a-b|$ when $|a|, |b| \leq 1$, we obtain

$$\left| \int_{|t|\leq T} \left(|\phi(t)|^s - \left|\frac{\sin|t|}{|t|}\right|^s \right) |t|^{-2} \mathrm{d}t \right| \leq s\frac{\delta(\delta+2)}{2} \int_{|t|\leq T} \mathrm{d}t = \frac{2\pi}{3} s\delta(\delta+2)T^3.$$

Large t. Since $s \geq 2$, we have

$$\left| \int_{|t|\geq T} \left(|\phi(t)|^s - \left|\frac{\sin|t|}{|t|}\right|^s \right) |t|^{-2} \mathrm{d}t \right| \leq \int_{|t|\geq T} \left(|\phi(t)|^2 + \left|\frac{\sin|t|}{|t|}\right|^2 \right) |t|^{-2} \mathrm{d}t.$$

By virtue of the decay assumption (5), this is at most

$$\int_{|t|\geq T} \frac{C_0^2+1}{|t|^4} \mathrm{d}t = 4\pi \frac{C_0^2+1}{T}$$

Adding up these two bounds and optimising over T yields

$$\left| \int_{\mathbb{R}^3} \left(|\phi(t)|^s - \left|\frac{\sin|t|}{|t|}\right|^s \right) |t|^{-2} \mathrm{d}t \right| \leq \frac{2^{15/4}\pi}{3} s^{1/4} \big(\delta(\delta+2)\big)^{1/4} \big(C_0^2+1\big)^{3/4}.$$

Plugging this back gives the assertion. □

4.2.2. *The Gaussian approximation.* We now present a bound on $\Phi(s)$ which does not grow as $s \to \infty$ that will allow us to prove Lemma 10 for s sufficiently large.

Lemma 13. *Let X be a symmetric random vector in $\mathbb{R}^3$ with $\delta = \mathsf{W}_2(X,\xi)$ and characteristic function ϕ satisfying* (5) *for some $C_0 > 0$. Let Φ be defined through* (22)*. Assuming that $\delta \leq \min\{\frac{1}{\sqrt{3}}, (15C_0)^{-2}\}$, we have*

$$\begin{aligned} \Phi(s) \leq & \sqrt{\frac{6}{\pi}} \Big((1-\delta\sqrt{3})^2 - \theta \mathbb{E}|X|^3 \Big)^{-1/2} \\ & + \sqrt{\frac{6}{\pi}} \exp\left\{ -s\left(\frac{\theta^2}{6} - 26\delta(\delta+2) \right) \right\} + 2C_0 \left(\sqrt{s} + \frac{2}{\sqrt{s}} \right) e^{-s}, \quad s \geq 2, \end{aligned} \tag{27}$$

with arbitrary $0 < \theta < \frac{(1-\delta\sqrt{3})^2}{3\mathbb{E}|X|^3}$.

Proof. We split the integral defining $\Phi(s) = \frac{1}{2\pi^2}\int_{\mathbb{R}^3} |\phi(s^{-1/2}t)|^s |t|^{-2} \mathrm{d}t$ into several regions.
Large t. Using the decay condition (5), we get

$$\int_{|t|\geq eC_0\sqrt{s}} \left| \phi\left(s^{-1/2}t\right) \right|^s |t|^{-2} \mathrm{d}t \leq \int_{|t|\geq eC_0\sqrt{s}} C_0^s |s^{-1/2}t|^{-s} |t|^{-2} \mathrm{d}t = \frac{4\pi e\sqrt{s}}{s-1} C_0 e^{-s}.$$

Thus, for $s \geq 2$,

$$\frac{1}{2\pi^2}\int_{|t|\geq eC_0\sqrt{s}} \left|\phi\left(s^{-1/2}t\right)\right|^s |t|^{-2}\mathrm{d}t \leq \frac{2e\sqrt{s}}{\pi(s-1)}C_0e^{-s} < \frac{4C_0}{\sqrt{s}}e^{-s},$$

as $\frac{2e\sqrt{s}}{\pi(s-1)} < \frac{4}{\sqrt{s}}$ for $s \geq 2$.

Moderate t. This case is vacuous unless $C_0 > \pi/e$. We use Lemma 11 to obtain

$$\begin{aligned}
&\int_{\pi\sqrt{s}\leq|t|\leq eC_0\sqrt{s}} \left|\phi\left(s^{-1/2}t\right)\right|^s |t|^{-2}\mathrm{d}t \\
&\leq \int_{\pi\sqrt{s}\leq|t|\leq eC_0\sqrt{s}} \left(\left|\frac{\sin(s^{-1/2}|t|)}{s^{-1/2}|t|}\right| + \frac{\delta(\delta+2)}{2}\left(s^{-1/2}|t|\right)^2\right)^s |t|^{-2}\mathrm{d}t \\
&\leq \int_{\pi\sqrt{s}\leq|t|\leq eC_0\sqrt{s}} \left(\frac{1}{\pi} + \frac{\delta(\delta+2)}{2}(eC_0)^2\right)^s |t|^{-2}\mathrm{d}t \\
&= 4\pi\sqrt{s}\left(\frac{1}{\pi} + \frac{\delta(\delta+2)}{2}(eC_0)^2\right)^s (eC_0-\pi)_+.
\end{aligned}$$

In this case, the condition $\delta < (15C_0)^{-2}$ suffices to guarantee that $\frac{1}{\pi} + \frac{\delta(\delta+2)}{2}(eC_0)^2 < \frac{1}{e}$ (also using, say $\delta + 2 < 3$). Then we get

$$\frac{1}{2\pi^2}\int_{\pi\sqrt{s}\leq|t|\leq eC_0\sqrt{s}} \left|\phi\left(s^{-1/2}t\right)\right|^s |t|^{-2}\mathrm{d}t \leq \frac{2}{\pi}\sqrt{s}e^{-s}(eC_0-\pi)_+ < 2C_0\sqrt{s}e^{-s}.$$

Small t. For $0 < u < \pi$, we have

$$\frac{\sin u}{u} = \prod_{k=1}^{\infty}\left(1 - \frac{u^2}{(k\pi)^2}\right) \leq \exp\Big(-\sum_{k=1}^{\infty}\frac{u^2}{(k\pi)^2}\Big) = e^{-u^2/6}. \tag{28}$$

Fix $0 < \theta < \pi$. Then, first using Lemma 11 and then (28), we obtain

$$\begin{aligned}
&\int_{\theta\sqrt{s}\leq|t|\leq\pi\sqrt{s}} \left|\phi\left(s^{-1/2}t\right)\right|^s |t|^{-2}\mathrm{d}t \\
&\leq \int_{\theta\sqrt{s}\leq|t|\leq\pi\sqrt{s}} \left(\left|\frac{\sin(s^{-1/2}|t|)}{s^{-1/2}|t|}\right| + \frac{\delta(\delta+2)}{2}\left(s^{-1/2}|t|\right)^2\right)^s |t|^{-2}\mathrm{d}t \\
&\leq \int_{\theta\sqrt{s}\leq|t|\leq\pi\sqrt{s}} \left(e^{-|t|^2/(6s)} + \frac{\delta(\delta+2)}{2}\pi^2\right)^s |t|^{-2}\mathrm{d}t \\
&\leq \int_{|t|\geq\theta\sqrt{s}} e^{-|t|^2/6}\left(1 + \frac{\delta(\delta+2)}{2}\pi^2e^{\pi^2/6}\right)^s |t|^{-2}\mathrm{d}t.
\end{aligned}$$

Integrating using polar coordinates and invoking the standard tail bound

$$\int_u^\infty e^{-y^2/2}\mathrm{d}y \leq \sqrt{\pi/2}e^{-u^2/2}, \qquad u > 0,$$

the last integral gets upper bounded by

$$4\pi^{3/2}\sqrt{\frac{3}{2}}e^{-\theta^2s/6}\left(1 + \frac{\delta(\delta+2)}{2}\pi^2e^{\pi^2/6}\right)^s < 4\pi^{3/2}\sqrt{\frac{3}{2}}e^{-\theta^2s/6}\big(1 + 26\delta(\delta+2)\big)^s,$$

Summarising, we have shown that

$$\frac{1}{2\pi^2}\int_{\theta\sqrt{s}\le|t|\le\pi\sqrt{s}}\left|\phi\left(s^{-1/2}t\right)\right|^s|t|^{-2}\mathrm{d}t\le\sqrt{\frac{6}{\pi}}\left(1+26\delta(\delta+2)\right)^s e^{-s\theta^2/6}$$
$$\le\sqrt{\frac{6}{\pi}}\exp\left\{-s\left(\frac{\theta^2}{6}-26\delta(\delta+2)\right)\right\}.$$

Very small t. Taylor-expanding ϕ at 0 with the Lagrange remainder,

$$\int_{|t|\le\theta\sqrt{s}}\left|\phi\left(s^{-1/2}t\right)\right|^s|t|^{-2}\mathrm{d}t$$
$$=\int_{|t|\le\theta\sqrt{s}}\left|1-\frac{1}{2}\mathbb{E}\left\langle X,s^{-1/2}t\right\rangle^2+\frac{s^{-3/2}}{6}\sum_{j,k,l=1}^{3}\frac{\partial^3\phi}{\partial t_j\partial t_k\partial t_l}(\theta)t_jt_kt_l\right|^s\mathrm{d}t,$$

for some point θ in the segment $[0,s^{-1/2}t]$. To bound the error term, we note that

$$\left|\frac{\partial^3\phi}{\partial t_j\partial t_k\partial t_l}(\theta)\right|\le\mathbb{E}|X_jX_kX_l|,$$

thus

$$\left|\sum_{j,k,l=1}^{3}\frac{\partial^3\phi}{\partial t_j\partial t_k\partial t_l}(\theta)t_jt_kt_l\right|\le\mathbb{E}\left(|t_1||X_1|+|t_2||X_2|+|t_3||X_3|\right)^3\le|t|^3\mathbb{E}|X|^3.$$

We also note that in the domain $\{|t|\le\theta\sqrt{s}\}$, the leading term $1-\frac{1}{2}\mathbb{E}\langle X,s^{-1/2}t\rangle^2$ is nonnegative, provided that $\frac{1}{2}\theta^2\mathbb{E}|X|^2\le 1$. Since $\|X\|_2\le\delta+1$ under the assumption (6), it suffices that $\theta<\frac{\sqrt{2}}{1+\delta}$. Assuming this, we thus get

$$\int_{|t|\le\theta\sqrt{s}}\left|\phi\left(s^{-1/2}t\right)\right|^s|t|^{-2}\mathrm{d}t\le\int_{|t|\le\theta\sqrt{s}}\left(1-\frac{1}{2}\mathbb{E}\left\langle X,s^{-1/2}t\right\rangle^2+\frac{1}{6}|s^{-1/2}t|^3\mathbb{E}|X|^3\right)^s|t|^{-2}\mathrm{d}t.$$

Evoking (6), let ξ be uniform on $\mathbb{S}^2$ such that $\|X-\xi\|_2\le\delta$ with respect to some coupling. Then, for a fixed vector v in $\mathbb{R}^3$, we obtain the bound

$$\|\langle X,v\rangle\|_2\ge\|\langle\xi,v\rangle\|_2-\|\langle X-\xi,v\rangle\|_2=\tfrac{1}{\sqrt{3}}|v|-\|\langle X-\xi,v\rangle\|_2\ge\tfrac{1}{\sqrt{3}}|v|-\delta|v|.$$

Thus, provided that $\delta<\frac{1}{\sqrt{3}}$, this yields

$$\int_{|t|\le\theta\sqrt{s}}\left|\phi\left(s^{-1/2}t\right)\right|^s|t|^{-2}\mathrm{d}t\le\int_{|t|\le\theta\sqrt{s}}\left(1-\frac{(1/\sqrt{3}-\delta)^2}{2s}|t|^2+\frac{\theta\mathbb{E}|X|^3}{6s}|t|^2\right)^s|t|^{-2}\mathrm{d}t$$
$$\le\int_{\mathbb{R}^3}\exp\left(-\alpha|t|^2/2\right)|t|^{-2}\mathrm{d}t=\frac{2\pi\sqrt{2\pi}}{\sqrt{\alpha}},$$

where we have set $\alpha=(\frac{1}{\sqrt{3}}-\delta)^2-\frac{1}{3}\theta\mathbb{E}|X|^3$ and assumed that α is positive in the last equality (guaranteed by choosing θ sufficiently small). Then we finally obtain

$$\frac{1}{2\pi^2}\int_{|t|\le\theta\sqrt{s}}\left|\phi\left(s^{-1/2}t\right)\right|^s|t|^{-2}\mathrm{d}t\le\sqrt{\frac{2}{\pi\alpha}}.$$

Putting these three bounds together gives the assertion. Note that we have imposed the conditions $\delta<\frac{1}{\sqrt{3}}$ and $\delta<(15C_0)^{-2}$ when $C_0>\frac{\pi}{e}$, as well as $\theta<\pi$, $\theta<\frac{\sqrt{2}}{1+\delta}$ and

$\theta < \frac{(1-\delta\sqrt{3})^2}{3\mathbb{E}|X|^3}$. Since $\|X\|_3 \geq \|X\|_2 \geq 1-\delta$ and $\delta < \frac{1}{\sqrt{3}}$, we have $\frac{(1-\delta\sqrt{3})^2}{3\mathbb{E}|X|^3} < \frac{(1-\delta\sqrt{3})^2}{3(1-\delta)^3} = \frac{1}{3(1-\delta)}\left(\frac{1-\delta\sqrt{3}}{1-\delta}\right)^2 < \frac{1}{3-\sqrt{3}} < 0.79$. Moreover, $\frac{\sqrt{2}}{1+\delta} > \frac{\sqrt{2}}{1+1/\sqrt{3}} > 0.89$, so the condition $\theta < \frac{(1-\delta\sqrt{3})^2}{3\mathbb{E}|X|^3}$ implies the other two conditions on θ. □

4.3. **Bounds on the derivative of the special function.**

Lemma 14. *Let X be a symmetric random vector in $\mathbb{R}^3$ with $\delta = \mathsf{W}_2(X,\xi)$ and characteristic function ϕ satisfying* (5) *for some $C_0 > 0$. Let Φ and Φ_0 be defined through* (22) *and* (24) *respectively. For every $s \geq 2$, we have*

$$|\Phi'(s) - \Phi_0'(s)| \leq \frac{2^{7/4}}{3\pi}\big(\delta(\delta+2)\big)^{1/4}\big(C_0^2+1\big)^{3/4}s^{-1/4} + 1.04\big(\delta(\delta+2)\big)^{1/7}\big(C_0^{3/2}+1\big)^{6/7}s^{1/2}.$$

Proof. First we take the derivative,

$$\Phi'(s) = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\sqrt{s}}{2\pi^2}\int_{\mathbb{R}^3}|\phi(t)|^s\mathrm{d}t\right) = \frac{1}{2s}\Phi(s) + \frac{\sqrt{s}}{2\pi^2}\int_{\mathbb{R}^3}|\phi(t)|^s\log|\phi(t)|\mathrm{d}t.$$

For the resulting $\Phi - \Phi_0$ term, we use Lemma 12. To bound the difference of the integrals resulting from the second term, we fix $T > 0$ and split the integration into two regions.
Small t. Using Lemmas 7 and 11, we obtain

$$\begin{aligned}&\left|\int_{|t|\leq T}\left(|\phi(t)|^s\log|\phi(t)| - \left|\frac{\sin|t|}{|t|}\right|^s\log\left|\frac{\sin|t|}{|t|}\right|\right)|t|^{-2}\mathrm{d}t\right|\\ &\leq \int_{|t|\leq T}\frac{\delta(\delta+2)}{2}\mathrm{d}t = \frac{2\pi}{3}\delta(\delta+2)T^3.\end{aligned}$$

Large t. Note that for $s \geq 2$, and $0 < u < 1$ we have,

$$|u^s\log u| = |2u^{s-1/2}u^{1/2}\log(u^{1/2})| \leq \frac{2}{e}u^{3/2}.$$

Thus,

$$\begin{aligned}&\left|\int_{|t|\geq T}\left(|\phi(t)|^s\log|\phi(t)| - \left|\frac{\sin|t|}{|t|}\right|^s\log\left|\frac{\sin|t|}{|t|}\right|\right)|t|^{-2}\mathrm{d}t\right|\\ &\leq \frac{2}{e}\left|\int_{|t|\geq T}\left(|\phi(t)|^{3/2} + \left|\frac{\sin|t|}{|t|}\right|^{3/2}\right)|t|^{-2}\mathrm{d}t\right|\end{aligned}$$

which, after applying the decay condition (5), gets upper bounded by

$$\frac{8\pi}{e}\int_T^\infty\frac{C_0^{3/2}+1}{t^{3/2}}\mathrm{d}t = \frac{16\pi}{e}(C_0^{3/2}+1)T^{-1/2}.$$

Adding up these two bounds and optimising over T yields

$$\begin{aligned}&\left|\int_{\mathbb{R}^3}\left(|\phi(t)|^s\log|\phi(t)| - \left|\frac{\sin|t|}{|t|}\right|^s\log\left|\frac{\sin|t|}{|t|}\right|\right)|t|^{-2}\mathrm{d}t\right|\\ &\leq \frac{7\cdot 2^{19/7}\pi}{3e^{6/7}}\big(\delta(\delta+2)\big)^{1/7}\big(C_0^{3/2}+1\big)^{6/7}.\end{aligned}$$

Going back to the difference of the derivatives, we arrive at the desired bound using

$$\frac{7\cdot 2^{12/7}}{3e^{6/7}\pi} < 1.04. \qquad \square$$

4.4. **Bounds on Ball's special function.** We will need two estimates on Φ_0 defined in (24), that is

$$\Phi_0(s) = \frac{2}{\pi}\int_0^\infty \left|\frac{\sin(s^{-1/2}t)}{s^{-1/2}t}\right|^s \mathrm{d}t = \frac{2\sqrt{s}}{\pi}\int_0^\infty \left|\frac{\sin t}{t}\right|^s \mathrm{d}t, \qquad s > 1. \tag{29}$$

First, we have a bound on the derivative near $s = 2$.

Lemma 15. *For* $2 \le s \le 2.01$*, we have* $\Phi_0'(s) \le -0.02$*.*

Second, on the complementary range, $\Phi_0(s)$ is separated from its supremal value $\Phi_0(2)$.

Lemma 16. *For* $s \ge 2.01$*, we have* $\Phi_0(s) \le \Phi_0(2) - 2\cdot 10^{-4}$*.*

We begin with a numerical bound which will be used in the proofs of these assertions.

Lemma 17. *We have*

$$\int_0^\infty \left(\frac{\sin u}{u}\right)^2 \log\left|\frac{\sin u}{u}\right| \mathrm{d}u \le -0.48.$$

Proof. Using (28), we get

$$\int_0^\pi \left(\frac{\sin u}{u}\right)^2 \log\left|\frac{\sin u}{u}\right| \mathrm{d}u \le -\frac{1}{6}\int_0^\pi (\sin u)^2 \mathrm{d}u = -\frac{\pi}{12}.$$

Moreover,

$$\begin{aligned}\int_\pi^\infty \left(\frac{\sin u}{u}\right)^2 \log\left|\frac{\sin u}{u}\right| \mathrm{d}u &= \sum_{k=1}^\infty \int_{k\pi}^{(k+1)\pi} \left(\frac{\sin u}{u}\right)^2 \log\left|\frac{\sin u}{u}\right| \\ &\le \sum_{k=1}^\infty \int_{k\pi}^{(k+1)\pi} \left(\frac{\sin u}{(k+1)\pi}\right)^2 \log\left|\frac{1}{k\pi}\right| = -\frac{1}{2\pi}\sum_{k=1}^\infty \frac{\log(k\pi)}{(k+1)^2}.\end{aligned}$$

Therefore our integral is bounded above by

$$-\frac{\pi}{12} - \frac{1}{2\pi}\sum_{k=1}^\infty \frac{\log(k\pi)}{(k+1)^2} = -0.4867.. < -0.48. \qquad \square$$

We let

$$I(s) = \int_0^\infty \left|\frac{\sin u}{u}\right|^s \mathrm{d}u, \qquad s > 1. \tag{30}$$

Proof of Lemma 15. First we observe that

$$I'(s) = \int_0^\infty \left|\frac{\sin u}{u}\right|^s \log\left|\frac{\sin u}{u}\right| \mathrm{d}u.$$

Note that I is decreasing. We have,

$$\Phi_0'(s) = \frac{2}{\pi}\left(\frac{I(s)}{2\sqrt{s}} + \sqrt{s}I'(s)\right) \le \frac{2}{\pi}\left(\frac{I(2)}{2\sqrt{s}} + \sqrt{s}I'(s)\right) = \frac{1}{2\sqrt{s}} + \frac{2\sqrt{s}}{\pi}I'(s),$$

since $I(2)=\frac{\pi}{2}$. Moreover,

$$\begin{aligned}|I''(s)| &= \int_0^\infty \left|\frac{\sin u}{u}\right|^s \log^2\left|\frac{\sin u}{u}\right| \mathrm{d}u \le \int_0^\infty \left|\frac{\sin u}{u}\right|^2 \log^2\left|\frac{\sin u}{u}\right| \mathrm{d}u\\ &\le \sup_{t\in(0,1)} (\sqrt{t}\log^2 t) \int_0^\infty \left|\frac{\sin u}{u}\right|^{3/2} \mathrm{d}u = 16e^{-2}\int_0^\infty \left|\frac{\sin u}{u}\right|^{3/2} \mathrm{d}u\\ &\le 16e^{-2}\left(1+\int_1^\infty \frac{1}{u^{3/2}}\mathrm{d}u\right) = 48e^{-2}.\end{aligned}$$

With the aid of Lemma 17, we therefore have

$$I'(s) \le I'(2) + 48e^{-2}(s-2) < -0.48 + 48e^{-2}(s-2).$$

Thus, for $2 \le s \le 2.01$, we have

$$\begin{aligned}\Phi_0'(s) &\le \frac{1}{2\sqrt{s}} + \frac{2\sqrt{s}}{\pi} I'(s) < \frac{1}{2\sqrt{s}} + \frac{2\sqrt{s}}{\pi}\big(-0.48 + 48e^{-2}(s-2)\big)\\ &< \frac{1}{2\sqrt{2}} + \frac{2\sqrt{2}}{\pi}(-0.48 + 48e^{-2}(s-2))\\ &\le \frac{1}{2\sqrt{2}} + \frac{2\sqrt{2}}{\pi}(-0.48 + 48e^{-2}0.01) < -0.02,\end{aligned}$$

where in the first inequality we used that the term in parenthesis is negative. ☐

For the proof of Lemma 16, we need several more estimates. First, we record a lower bound on the derivative of $\Phi_0(s)$ for arbitrary s.

Lemma 18. *For $s \ge 2$, we have $\Phi_0'(s) \ge -\frac{12\sqrt{s}}{\pi e}$.*

Proof. We have,

$$\Phi_0'(s) = \frac{2}{\pi}\left(\frac{I(s)}{2\sqrt{s}} + \sqrt{s}I'(s)\right) \ge \frac{2\sqrt{s}}{\pi} I'(s),$$

so it is enough to upper bound $|I'(s)|$. Note that

$$\begin{aligned}|I'(s)| &= \int_0^\infty \left|\frac{\sin u}{u}\right|^s \left(-\log\left|\frac{\sin u}{u}\right|\right) \mathrm{d}u\\ &\le \int_0^\infty \left|\frac{\sin u}{u}\right|^2 \left(-\log\left|\frac{\sin u}{u}\right|\right) \mathrm{d}u\\ &\le \sup_{t\in(0,1)} (-\sqrt{t}\log t) \int_0^\infty \left|\frac{\sin u}{u}\right|^{\frac{3}{2}} \mathrm{d}u\\ &\le 2e^{-1}\left(1+\int_1^\infty \frac{1}{u^{\frac{3}{2}}}\mathrm{d}u\right) = 6e^{-1}.\end{aligned}$$ ☐

Second, we obtain a quantitative drop-off of the values of Φ_0.

Lemma 19. *Let $a \in [1, \frac{\pi}{3}]$ and suppose that for some $s_0 \ge 2$, we have $\Phi_0(s_0) = \sqrt{\frac{2}{a}}$. Then*

$$\Phi_0(s) \le \sqrt{\frac{2}{a}}, \qquad s \ge s_0. \tag{31}$$

To prove this, we build on the argument of Nazarov and Podkorytov from [31]. For a somewhat similar bound, we refer to Proposition 7 in König and Koldobsky's work [20] on maximal-perimeter sections of the cube. For convenience and completeness, we include all arguments in detail. We consider functions

$$f_a(x) = e^{-\frac{\pi}{2}x^2 a}, \quad g(x) = \left|\frac{\sin \pi x}{\pi x}\right|, \qquad x > 0, \tag{32}$$

and their distribution functions

$$F_a(y) = |\{x > 0 : f_a(x) > y\}|, \quad G(y) = |\{x > 0 : g(x) > y\}|, \qquad y > 0. \tag{33}$$

Lemma 20. *For $a \in [1, \frac{\pi}{3}]$ the function $F_a - G$ has precisely one sign change point y_0 and at this point changes sign from " $-$ " to " $+$ ".*

Proof. Note that $F_a(y) = G(y) = 0$ for $y \geq 1$, so we only consider $y \in (0,1)$. We have $F_a(y) = \sqrt{\frac{2}{\pi a}\ln(\frac{1}{y})}$.

The function $g(x)$ has zeros for $x \in \mathbb{Z}$. For $m \in \mathbb{N}$, let $y_m = \max_{[m,m+1]} g$. We clearly have $y_m < \frac{1}{\pi m}$ and $y_m > g(m+\frac{1}{2}) = \frac{1}{\pi(m+\frac{1}{2})}$. Thus $y_m \in (\frac{1}{\pi(m+\frac{1}{2})}, \frac{1}{\pi m})$, which shows that the sequence y_m is decreasing. We have the following claims.

Claim 1. The function $F_a - G$ is positive on $(y_1, 1)$.

Note that if $g(x) > y_1$ then $x \in (0,1)$. Moreover $g(x) \leq f(x)$ for $x \in [0,1]$, since

$$g(x) = \frac{\sin \pi x}{\pi x} = \prod_{k=1}^{\infty}\left(1 - \frac{x^2}{k^2}\right) \leq \prod_{k=1}^{\infty} e^{-\frac{x^2}{k^2}} = e^{-\frac{\pi^2}{6}x^2} \leq e^{-\frac{\pi}{2}ax^2} = f_a(x).$$

Thus, for $y \in (y_1, 1)$, we have

$$G(y) = |\{x \in (0,1) : \ g(x) > y\}| < |\{x \in (0,1) : \ f_a(x) > y\}| \leq F_a(y).$$

Claim 2. The function $F_a - G$ changes sign at least once in $(0,1)$.

Due to Claim 1 it is enough to show that $F_a - G$ is sometimes negative. We have $F_a - G \leq F_1 - G$ and $\int_0^\infty 2y(F_1(y) - G(y))\mathrm{d}y = \int (f_1^2 - g^2) = 0$, so $F_1 - G$ can be negative.

Claim 3. The function $F_a - G$ is increasing on $(0, y_1)$.

Clearly $F_a' > F_1'$ and thus the claim follows from the fact that $F_1 - G$ is increasing on $(0, y_1)$, which was proved in [31] (Chapter I, Step 5). □

Proof of Lemma 19. The assumption $\Phi_0(s_0) = \sqrt{\frac{2}{\alpha}}$ is equivalent to

$$\int_0^\infty \left|\frac{\sin \pi x}{\pi x}\right|^{s_0} \mathrm{d}x = \int_0^\infty \left|e^{-\frac{\pi}{2}x^2 a}\right|^{s_0} \mathrm{d}x.$$

After changing variables and using Lemma 20, we get from the Nazarov–Podkorytov lemma (Chapter I, Step 4 in [31]) that for $s \geq s_0$

$$\int_0^\infty \left|\frac{\sin x}{x}\right|^s \mathrm{d}x \leq \int_0^\infty \left|e^{-\frac{1}{2\pi}x^2 a}\right|^s \mathrm{d}x = \frac{\pi}{\sqrt{2as}}. \qquad \square$$

Proof of Lemma 16. Take $s_0 = 2.01$ and $a = 2\Phi_0(s_0)^{-2}$ in Lemma 19. Since $\Phi_0(2) = \sqrt{2}$, Ball's inequality gives that $a \geq 1$. We need to check that $a \leq \frac{\pi}{3}$. From Lemma 18, we have that for $s \in [2, 2.01]$, $\Phi_0'(s) \geq -\frac{12\sqrt{2.01}}{\pi e} > -2$. Thus, $\Phi_0(s_0) \geq \Phi_0(2) - 2(s_0 - 2) = \sqrt{2} - 0.02$. Therefore, $a < 2 \cdot (\sqrt{2} - 0.02)^{-2} < 1.03 < \frac{\pi}{3}$, as needed. By Lemmas 19 and 15, we thus get that for $s \geq s_0 = 2.01$,

$$\Phi_0(s) \leq \sqrt{\frac{2}{a}} = \Phi_0(s_0) \leq \Phi_0(2) + \sup_{[2,2.01]} \Phi_0' \cdot 0.01 \leq \Phi_0(2) - 0.02 \cdot 0.01. \qquad \square$$

4.5. **Proof of Lemma 10.** Recall that we assume X is a symmetric random vector in $\mathbb{R}^3$ with $\delta = \mathsf{W}_2(X, \xi)$ and characteristic function ϕ satisfying (5), that is $|\phi(t)| \leq C_0/|t|$, for all $t \in \mathbb{R}^3 \setminus \{0\}$. Let $C_1 = \max\{C_0, 1\}$. Our goal is to show that if (6) holds, that is

$$\delta \leq 10^{-38} C_1^{-9} \min\left\{(\mathbb{E}|X|^3)^{-6}, 1\right\},$$

then $\Phi(s) \leq \Phi(2)$ for all $s \geq 2$, where Φ is defined in (22). For the sake of clarity, we shall be fairly lavish with choosing constants. Since $C_1 \geq 1$, the above assumes in particular that $\delta \leq 10^{-38}$. With this in mind, we note the following consequences of Lemmas 12 and 14 respectively: for $s \geq 2$,

$$|\Phi(s) - \Phi_0(s)| \leq \frac{2^{11/4}}{3\pi} s^{3/4} \big(\delta(\delta+2)\big)^{1/4} \big(C_0^2 + 1\big)^{3/4} < 2 s^{3/4} \delta^{1/4} C_1^{3/2} \tag{34}$$

and similarly

$$|\Phi'(s) - \Phi_0'(s)| < s^{-1/4} \delta^{1/4} C_1^{3/2} + 2.1 \cdot s^{1/2} \delta^{1/7} C_1^{9/7}. \tag{35}$$

We also remark that $\|X\|_3 \geq \|X\|_2 \geq \|\xi\|_2 - \|X - \xi\|_2 = 1 - \delta \geq 1 - 10^{-38}$.

We break the argument into several regimes for the parameter s.

Large s. With hindsight, we set

$$s_0 = \max\left\{10^6 (\mathbb{E}|X|^3)^2, 2\log C_1\right\} \tag{36}$$

In particular, $s_0 \geq 10^5$. Using Lemma 13, that is

$$\Phi(s) \leq \sqrt{\frac{6}{\pi}} \Big((1 - \delta\sqrt{3})^2 - \theta \mathbb{E}|X|^3\Big)^{-1/2} + \sqrt{\frac{6}{\pi}} \exp\left\{-s\left(\frac{\theta^2}{6} - 26\delta(\delta+2)\right)\right\} + 2C_0 \left(\sqrt{s} + \frac{2}{\sqrt{s}}\right) e^{-s} = A_1 + A_2 + A_3,$$

we will show that $\Phi(s) \leq \Phi(2)$ for all $s \geq s_0$. We take $\theta = \frac{1}{100\mathbb{E}|X|^3}$ which satisfies the conditions of the lemma and then, for the first term A_1, we use

$$A_1 = \sqrt{\frac{6}{\pi}} \Big((1 - \delta\sqrt{3})^2 - \theta \mathbb{E}|X|^3\Big)^{-1/2} \leq \sqrt{\frac{6}{\pi}} \big(1 - 0.01\big)^{-1/2} < \sqrt{2} - \frac{1}{50}.$$

Thanks to (34), we also have

$$\sqrt{2} = \Phi_0(2) \leq \Phi(2) + 2^{7/4} \delta^{1/4} C_1^{3/2} = \Phi(2) + A_4,$$

so it suffices to show that each of the second and third terms A_2, A_3 as well as this additional error A_4 do not exceed $\frac{1}{150}$. Using $\delta < 10^{-38} C_1^{-9}$, we get

$$A_4 \leq 2^{7/4} \cdot 10^{-19/2} C_1^{-3/4} < \frac{1}{150}.$$

For the exponent in the second term A_2, observe that

$$26\delta(\delta+2) < 53\delta < 53\cdot 10^{-38}C_1^{-9}(\mathbb{E}|X|^3)^{-6} \leq 10^{-36}(\mathbb{E}|X|^3)^{-2},$$

and, consequently,

$$\frac{\theta^2}{6} - 26\delta(\delta+2) \geq \frac{1}{6\cdot 10^4(\mathbb{E}|X|^3)^2} - \frac{1}{10^{36}(\mathbb{E}|X|^3)^2} \geq \frac{1}{10^5(\mathbb{E}|X|^3)^2}.$$

Thus, using $s \geq s_0 \geq 10^6(\mathbb{E}|X|^3)^2$, we get

$$A_2 \leq \sqrt{\frac{6}{\pi}}\exp\left\{-\frac{s_0}{10^5(\mathbb{E}|X|^3)^2}\right\} \leq \sqrt{\frac{6}{\pi}}\exp\{-10\} < \frac{1}{150}.$$

Finally, for the third term, since $s \geq s_0 \geq 10^5$,

$$\left(\sqrt{s}+\frac{2}{\sqrt{s}}\right)e^{-s} \leq (\sqrt{s}+1)e^{-s} \leq e^{\sqrt{s}-s} \leq \frac{1}{300}e^{-s/2},$$

therefore, since $s \geq s_0 \geq 2\log C_1$,

$$A_3 \leq 2C_1\left(\sqrt{s}+\frac{2}{\sqrt{s}}\right)e^{-s} \leq \frac{C_1}{150}e^{-s/2} \leq \frac{1}{150}.$$

Moderate s. We now assume that $2.01 \leq s \leq s_0$. Using (34) twice and Lemma 16,

$$\begin{aligned}\Phi(s) \leq \Phi_0(s) + 2s_0^{3/4}\delta^{1/4}C_1^{3/2} &\leq \Phi_0(2) - 2\cdot 10^{-4} + 2s_0^{3/4}\delta^{1/4}C_1^{3/2} \\ &\leq \Phi(2) - 2\cdot 10^{-4} + 2\cdot 2^{3/4}\delta^{1/4}C_1^{3/2} + 2s_0^{3/4}\delta^{1/4}C_1^{3/2} \\ &\leq \Phi(2) - 2\cdot 10^{-4} + 3s_0^{3/4}\delta^{1/4}C_1^{3/2}.\end{aligned}$$

Inserting the bound on δ,

$$3s_0^{3/4}\delta^{1/4}C_1^{3/2} \leq 3\cdot 10^{-19/2}C_1^{-3/4}s_0^{3/4}\cdot\min\left\{(\mathbb{E}|X|^3)^{-3/2}, 1\right\}$$

If $s_0 = 10^6(\mathbb{E}|X|^3)^2$, then using the $(\mathbb{E}|X|^3)^{-3/2}$ term in the minimum and $C_1^{-3/4} \leq 1$, we get the above bounded by $3\cdot 10^{-19/2+9/2} = 3\cdot 10^{-5}$. If $s_0 = 2\log C_1$, then using the other term in the minimum, we get the bound by $3\cdot 2^{3/4}10^{-19/2}C_1^{-3/4}(\log C_1)^{3/4} < 3(2/e)^{3/4}10^{-19/2} < 10^{-4}$ since $u^{-1}\log u \leq e^{-1}$ for $u > 1$. In either case, we get the conclusion $\Phi(s) \leq \Phi(2)$.

Small s. We finally assume that $2 \leq s \leq 2.01$. To argue that $\Phi(s) \leq \Phi(2)$, we will show that $\Phi'(s) < 0$. By virtue of (35) and Lemma 15,

$$\begin{aligned}\Phi'(s) &\leq \Phi_0'(s) + s^{-1/4}\delta^{1/4}C_1^{3/2} + 2.1\cdot s^{1/2}\delta^{1/7}C_1^{9/7} \\ &< -0.02 + (\delta C_1^6)^{1/4} + 3\big(\delta C_1^9\big)^{1/7}.\end{aligned}$$

Since $\delta C_1^6 \leq \delta C_1^9 \leq 10^{-38}$, this is clearly negative and the proof is complete. □

5. Concluding remarks

Remark 1. Assumption (3) seems natural: plainly, there are distributions which are *not* close to the Rademacher one, for which the unit vector attaining $\inf\mathbb{E}|\sum a_jX_j|$ is different than $a = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \dots, 0)$, for instance it is $a = (1, 0, \dots, 0)$ for Gaussian mixtures (see

[1, 10]), or for the Rademacher distribution with a large atom at 0 (see Theorem 4 and Remark 14 in [16]).

Remark 2. Handling the complementary case $\|a\|_\infty > \frac{1}{\sqrt{2}}$ which is not covered by Theorems 1 and 2 is a different story. The trivial convexity argument presented in the introduction works in fact only for the Rademacher case, as it requires $\frac{1}{\sqrt{2}}\mathbb{E}|X_1| \geq \mathbb{E}\left|\frac{X_1+X_2}{\sqrt{2}}\right|$, and only for the L_1-norm (see Remark 21 in [6]). To circumvent this, several different approaches have been used: Haagerup's ad hoc approximation (see §3 in [14]), Nazarov and Podkorytov's induction with a strengthened hypothesis (see Ch. II, Step 5 in [31]) which has also been adapted to other distributions (see [6, 5, 8]), and very recently a different inductive scheme near the extremiser (without a strengthening) needed in a geometric context (see [12]). None of these techniques appears amenable to the broad setting of general distributions that is treated in this paper.

Remark 3. De, Diakonikolas and Servedio obtained in [9] a stable version of Szarek's inequality (1) with respect to the unit vector a, namely

$$\mathbb{E}\left|\sum_{j=1}^n a_j\varepsilon_j\right| \geq \mathbb{E}\left|\frac{\varepsilon_1+\varepsilon_2}{\sqrt{2}}\right| + \kappa\sqrt{\delta(a)} \tag{37}$$

for a universal positive constant κ, where the deficit is given by $\delta(a) = |a-(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \dots, 0)|^2$, assuming that $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$. Note that in the setting of Theorem 1, we have

$$\left|\mathbb{E}\left|\sum_{j=1}^n a_jX_j\right| - \mathbb{E}\left|\sum_{j=1}^n a_j\varepsilon_j\right|\right| \leq \delta_0,$$

by a simple application of the triangle inequality and $\|\cdot\|_1 \leq \|\cdot\|_2$. Thus, applying this (twice) and the bound (37) of De Diakonikolas and Servedio, we conclude that Theorem 1 also holds for unit vectors a with $\delta(a) \geq (2\delta_0/\kappa)^2$. The same will apply to Theorem 2 with the aid of Theorem 1.2 from [7], a strengthening of Ball's inequality (2) (see also [27]). See [12] for numerical values of the constants κ.

Remark 4. We have used the W_2-distance in Theorems 1 and 2 for concreteness and convenience. Of course, for every $p \geq 1$, if we use the W_p-distance in (3) and assume that X_1 is in $L_{\frac{p}{p-1}}$, then the proofs of Lemmas 5 and 11 go through with the Cauchy–Schwarz inequality replaced by Hölder's inequality and the rest of the proof remains unchanged. It might be of interest to examine weaker distances in such statements.

References

[1] Roland Averkamp and Christian Houdré. Wavelet thresholding for non-necessarily Gaussian noise: Idealism. Ann. Statist. 31 (2003), 110–151.

[2] Keith Ball. Cube slicing in $\mathbf{R}^n$. Proc. Amer. Math. Soc. 97 (1986), no. 3, 465–473.

[3] Franck Barthe and Assaf Naor. Hyperplane projections of the unit ball of ℓ_p^n. Discrete Comput. Geom. 27 (2002), no. 2, 215–226.

[4] Albert Baernstein II and Robert Culverhouse. Majorization of sequences, sharp vector Khinchin inequalities, and bisubharmonic functions. Studia Math. 152 (2002), no. 3, 231–248.

[5] Giorgos Chasapis, Keerthana Gurushankar and Tomasz Tkocz. Sharp bounds on p-norms for sums of independent uniform random variables, $0 < p < 1$. To appear in J. Anal. Math., preprint (2021), arXiv:2105.14079.

[6] Giorgos Chasapis, Hermann König and Tomasz Tkocz. From Ball's cube slicing inequality to Khinchin-type inequalities for negative moments. J. Funct. Anal. 281 (2021), no. 9, Paper No. 109185, 23 pp.

[7] Giorgos Chasapis, Piotr Nayar and Tomasz Tkocz. Slicing ℓ_p-balls reloaded: Stability, planar sections in ℓ_1. Ann. Probab. 50 (2022), no. 6, 2344–2372.

[8] Giorgos Chasapis, Salil Singh and Tomasz Tkocz. Haagerup's phase transition at polydisc slicing. Preprint (2022), arXiv:2206.01026.

[9] Anindya De, Ilias Diakonikolas and Rocco A. Servedio. A robust Khintchine inequality, and algorithms for computing optimal constants in Fourier analysis and high-dimensional geometry. SIAM J. Discrete Math. 30 (2016), no. 2, 1058–1094.

[10] Alexandros Eskenazis, Piotr Nayar and Tomasz Tkocz. Gaussian mixtures: entropy and geometric inequalities. Ann. of Prob. 46(5) 2018, 2908–2945.

[11] Alexandros Eskenazis, Piotr Nayar and Tomasz Tkocz. Sharp comparison of moments and the log-concave moment problem. Adv. Math. 334 (2018), 389–416.

[12] Alexandros Eskenazis, Piotr Nayar and Tomasz Tkocz. Resilience of cube slicing in ℓ_p. Preprint available at arXiv:2211.01986 (2022).

[13] Evgeniĭ A. Gorin and Sergey Yu. Favorov. Generalizations of the Khinchin inequality. (Russian) Teor. Veroyatnost. i Primenen. 35 (1990), no. 4, 762–767; translation in Theory Probab. Appl. 35 (1990), no. 4, 766–771 (1991).

[14] Uffe Haagerup. The best constants in the Khintchine inequality. Studia Math. 70 (1981), no. 3, 231–283.

[15] Richard R. Hall. On a conjecture of Littlewood. Math. Proc. Cambridge Philos. Soc. 78 (1975), no. 3, 443–445.

[16] Alex Havrilla and Tomasz Tkocz. Sharp Khinchin-type inequalities for symmetric discrete uniform random variables. Israel J. Math. 246 (2021), no. 1, 281–297.

[17] Alex Havrilla, Piotr Nayar and Tomasz Tkocz. Khinchin-type inequalities via Hadamard's factorisation. To appear in Int. Math. Res. Not. IMRN, preprint (2021), arXiv:2102.09500.

[18] Hermann König. On the best constants in the Khintchine inequality for Steinhaus variables. Israel J. Math. 203 (2014), no. 1, 23–57.

[19] Hermann König and Alexander Koldobsky. On the maximal measure of sections of the n-cube. Geometric analysis, mathematical relativity, and nonlinear partial differential equations, 123–155, Contemp. Math., 599, Amer. Math. Soc., Providence, RI, 2013.

[20] Hermann König and Alexander Koldobsky. On the maximal perimeter of sections of the cube. Adv. Math. 346 (2019), 773–804.

[21] Hermann König and Stanisław Kwapień. Best Khintchine type inequalities for sums of independent, rotationally invariant random vectors. Positivity 5 (2001), no. 2, 115–152.

[22] Aleksandr Khintchine. Über dyadische Brüche. Math. Z. 18 (1923), no. 1, 109–116.

[23] Stanisław Kwapień, Rafał Latała and Krzysztof Oleszkiewicz. Comparison of moments of sums of independent random variables and differential inequalities. J. Funct. Anal. 136 (1996), no. 1, 258–268.

[24] Rafał Latała and Krzysztof Oleszkiewicz. On the best constant in the Khinchin-Kahane inequality. Studia Math. 109 (1994), no. 1, 101–104.

[25] Rafał Latała and Krzysztof Oleszkiewicz. A note on sums of independent uniformly distributed random variables. Colloq. Math. 68 (1995), no. 2, 197–206.

[26] John E. Littlewood. On bounded bilinear forms in an infinte number of variables. Quart. J. Math. Oxford Ser. 1 (1930), 164–174.

[27] James Melbourne and Cyril Roberto. Quantitative form of Ball's cube slicing in $\mathbb{R}^n$ and equality cases in the min-entropy power inequality. Proc. Amer. Math. Soc. 150 (2022), no. 8, 3595–3611.

[28] James Melbourne and Cyril Roberto. Transport-majorization to analytic and geometric inequalities. J. Funct. Anal. 284 (2023), no. 1, Paper No. 109717.

[29] Piotr Nayar and Krzysztof Oleszkiewicz. Khinchine type inequalities with optimal constants via ultra log-concavity. Positivity 16 (2012), no. 2, 359–371.

[30] Piotr Nayar and Tomasz Tkocz. Extremal sections and projections of certain convex bodies: a survey. Preprint (2022), arXiv:2210.00885.

[31] Fedor L. Nazarov and Anatoliy N. Podkorytov. Ball, Haagerup, and distribution functions. Complex analysis, operators, and related topics, 247–267, Oper. Theory Adv. Appl., 113, Birkhäuser, Basel, 2000.

[32] Charles M. Newman. An extension of Khintchine's inequality. Bull. Amer. Math. Soc. 81 (1975), no. 5, 913–915.

[33] Krzysztof Oleszkiewicz. Comparison of moments via Poincaré-type inequality. Advances in stochastic inequalities (Atlanta, GA, 1997), 135–148, Contemp. Math., 234, Amer. Math. Soc., Providence, RI, 1999.

[34] Stanisław Szarek. On the best constant in the Khintchine inequality. Stud. Math. 58, 197–208 (1976).

[35] Bogusław Tomaszewski. A simple and elementary proof of the Kchintchine inequality with the best constant. Bull. Sci. Math. (2) 111 (1987), no. 1, 103–109.

(A. E.) CNRS, Institut de Mathématiques de Jussieu, Sorbonne Université, France and Trinity College, University of Cambridge, UK.
Email address: `alexandros.eskenazis@imj-prg.fr, ae466@cam.ac.uk`

(P. N.) University of Warsaw, 02-097 Warsaw, Poland.
Email address: `nayar@mimuw@edu.pl`

(T. T.) Carnegie Mellon University, Pittsburgh, PA 15213, USA.
Email address: `ttkocz@andrew.cmu.edu`