

# TAIL BOUNDS FOR SUMS OF INDEPENDENT TWO-SIDED EXPONENTIAL RANDOM VARIABLES

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ABSTRACT. We establish upper and lower bounds with matching leading terms for tails of weighted sums of two-sided exponential random variables. This extends Janson's recent results for one-sided exponentials.

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*Key words.* exponential distribution, gamma distribution, concentration, tail bounds, sums of independent random variables.

## 1. INTRODUCTION

Concentration inequalities establish conditions under which random variables are *close* to their *typical values* (such as the expectation or median) and provide quantitative probabilistic bounds. Their significance cannot be overestimated, both across probability theory and in applications in related areas (see [1, 2]). Particularly, such inequalities often concern sums of independent random variables.

Let  $X_1, \dots, X_n$  be independent exponential random variables, each with mean 1. Consider their weighted sum  $S = \sum_{i=1}^n a_i X_i$  with some positive weights  $a_1, \dots, a_n$ . Janson in [11] showed the following concentration inequality: for every  $t > 1$ ,

$$(1) \quad \frac{1}{2e\alpha} \exp(-\alpha(t-1)) \leq \mathbb{P}(S \geq t\mathbb{E}S) \leq \frac{1}{t} \exp(-\alpha(t-1-\log t)),$$

where  $\alpha = \frac{\mathbb{E}S}{\max_{i \leq n} a_i}$  (in fact, he derived (1) from its analogue for the geometric distribution). Note that as  $t \rightarrow \infty$ , the lower and upper bounds are of the same order  $e^{-\alpha t + o(t)}$ . Moreover,  $e^{-\alpha t} = \mathbb{P}\left(X_1 > t \frac{\mathbb{E}S}{\max_{i \leq n} a_i}\right)$ . In words, the asymptotic behaviour of the tail of the sum  $S$  is the same as that of one summand carrying the largest weight.

The goal of this short note is to exhibit that the same behaviour holds for sums of two-sided exponentials (Laplace). Our main result reads as follows.

**Theorem 1.** *Let  $X_1, \dots, X_n$  be independent standard two-sided exponential random variables (i.e. with density  $\frac{1}{2}e^{-|x|}$ ,  $x \in \mathbb{R}$ ). Let  $S = \sum_{i=1}^n a_i X_i$  with  $a_1, \dots, a_n$  positive. For every  $t > 1$ ,*

$$(2) \quad \frac{1}{57\sqrt{\alpha t}} \exp(-\alpha t) \leq \mathbb{P}\left(S > t\sqrt{\text{Var}(S)}\right) \leq \exp\left(-\frac{\alpha^2}{2} h\left(\frac{2t}{\alpha}\right)\right),$$

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where  $\alpha = \frac{\sqrt{\text{Var}(S)}}{\max_{i \leq n} a_i} = \frac{\sqrt{2 \sum_{i=1}^n a_i^2}}{\max_{i \leq n} a_i}$ ,  $h(u) = \sqrt{1+u^2} - 1 - \log \frac{1+\sqrt{1+u^2}}{2}$ ,  $u > 0$ .

In (2), as  $t \rightarrow \infty$ , the lower and the upper bounds are of the same order,  $e^{-\alpha t + o(t)}$  (plainly,  $h(u) = u + o(u)$ ).

Our proof of Theorem 1 presented in Section 2 is based on an observation that two-sided exponentials are Gaussian mixtures, allowing to leverage (1) (this idea has recently found numerous uses in convex geometry, see [4, 5, 15]). In Section 3, we provide further generalisations of Janson's inequality (1) to certain nonnegative distributions, which also allows to extend Theorem 1 to a more general framework. We finish in Section 4 with several remarks (for instance, we deduce from (2) a formula for moments of  $S$ ).

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## 2. PROOF OF THEOREM 1

For the upper bound, we begin with a standard Chernoff-type calculation. Denote  $\sigma = \sqrt{\text{Var}(S)} = \sqrt{2 \sum a_i^2}$ . For  $\theta > 0$ , we have

$$\mathbb{P}(S \geq t\sigma) \leq e^{-\theta t\sigma} \mathbb{E} e^{\theta S}$$

and

$$\mathbb{E} e^{\theta S} = \prod \mathbb{E} e^{\theta a_i X_i} = \prod \frac{1}{1 - \theta^2 a_i^2} = \exp \left\{ - \sum \log(1 - \theta^2 a_i^2) \right\},$$

for  $\theta < \frac{1}{a_*}$ ,  $a_* = \max_{i \leq n} a_i$ . By convexity,

$$- \sum \log(1 - \theta^2 a_i^2) \leq - \sum \frac{a_i^2}{a_*^2} \log(1 - \theta^2 a_*^2),$$

so changing  $\theta$  to  $\theta/a_*$ , for every  $0 < \theta < 1$ , we have

$$\mathbb{P}(S \geq t\sigma) \leq \exp \left\{ -\theta t\alpha - \frac{\alpha^2}{2} \log(1 - \theta^2) \right\} = \exp \left\{ -\frac{\alpha^2}{2} \left( \frac{2t}{\alpha} \theta + \log(1 - \theta^2) \right) \right\},$$

where  $\alpha = \frac{\sigma}{a_*}$ . Optimising over  $\theta$  and using

$$\sup_{\theta \in (0,1)} \left( \theta u + \log(1 - \theta^2) \right) = \sqrt{1+u^2} - 1 - \log \frac{1+\sqrt{1+u^2}}{2}, \quad u > 0$$

gives the upper bound in (2) and thus finishes the argument.

For the lower bound, we shall use that a standard two-sided exponential random variable with density  $\frac{1}{2}e^{-|x|}$ ,  $x \in \mathbb{R}$ , has the same distribution as  $\sqrt{2}YG$ , where  $Y$  is an exponential random variable with mean 1 and  $G$  is a standard Gaussian random variable independent of  $Y$  (this follows for instance by checking that the characteristic functions are the same; see also a remark following Lemma 23 in [5]). This and the fact that sums of independent Gaussians are Gaussian justify the following claim, central to our argument.

**Proposition 2.** *The sum  $S = \sum_{i=1}^n a_i X_i$  has the same distribution as  $(2 \sum_{i=1}^n a_i^2 Y_i)^{1/2} G$  with  $Y_1, \dots, Y_n$  being independent mean 1 exponential random variables, independent of the standard Gaussian  $G$ .*

Recall  $\alpha = \frac{\sigma}{\max a_i}$ . Fix  $t > 1$ . By Proposition 2, for  $\theta > 0$ , we have

$$\begin{aligned}\mathbb{P}(S \geq t\sigma) &= \mathbb{P}\left(\sqrt{2 \sum a_i^2 Y_i} G \geq t\sigma\right) \geq \mathbb{P}\left(\sqrt{2 \sum a_i^2 Y_i} \geq \sqrt{\theta t \sigma^2}, G \geq \sqrt{\theta^{-1} t}\right) \\ &= \mathbb{P}\left(\sum a_i^2 Y_i \geq \frac{1}{2} \theta t \sigma^2\right) \mathbb{P}\left(G \geq \sqrt{\theta^{-1} t}\right).\end{aligned}$$

*Case 1.*  $t \geq \alpha$ . With hindsight, choose  $\theta = \frac{1}{\alpha}$ . Applying (1) to the first term yields

$$\mathbb{P}\left(\sum a_i^2 Y_i \geq \frac{1}{2} \theta t \sigma^2\right) = \mathbb{P}\left(\sum a_i^2 Y_i \geq \frac{t}{\alpha} \sum a_i^2\right) \geq \frac{1}{e\alpha^2} \exp\left\{-\frac{\alpha^2}{2} \left(\frac{t}{\alpha} - 1\right)\right\}.$$

For the second term we use a standard bound on the Gaussian tail,

$$(3) \quad \begin{aligned}\mathbb{P}(G > u) &\geq \frac{1}{\sqrt{2\pi}} \frac{u}{u^2 + 1} e^{-u^2/2}, & u > 0, \\ &\geq \frac{1}{2\sqrt{2\pi}} \frac{1}{u} e^{-u^2/2}, & u \geq 1.\end{aligned}$$

and as  $\theta^{-1}t = \alpha t \geq \sqrt{2}$ , (3) applies in our case. Combining the above estimates gives

$$\mathbb{P}(S \geq t\sigma) \geq \frac{\exp(\alpha^2/2)}{2\sqrt{2\pi}e\alpha^2} \frac{1}{\sqrt{\alpha t}} \exp(-\alpha t) \geq \frac{1}{4\sqrt{2\pi}} \frac{1}{\sqrt{\alpha t}} \exp(-\alpha t),$$

where in the last inequality we use that  $\inf_{x>1} \frac{1}{x} e^{x/2} = \frac{e}{2}$ .

*Case 2.*  $t \leq \alpha$ . With hindsight, choose  $\theta = \frac{1}{t}$ . Then

$$\mathbb{P}\left(\sum a_i^2 Y_i \geq \frac{1}{2} \theta t \sigma^2\right) = \mathbb{P}\left(\sum a_i^2 Y_i \geq \sum a_i^2\right).$$

To further lower-bound the last expression, we use a standard Paley-Zygmund type inequality (see, e.g. Lemma 3.2 in [17]).

**Lemma 3.** *Let  $Z_1, \dots, Z_n$  be independent mean 0 random variables such that  $\mathbb{E}Z_i^4 \leq C(\mathbb{E}Z_i^2)^2$  for all  $1 \leq i \leq n$  for some constant  $C \geq 1$ . Then for  $Z = Z_1 + \dots + Z_n$ ,*

$$\mathbb{P}(Z \geq 0) \geq \frac{1}{16^{1/3} \max\{C, 3\}}.$$

*Proof.* We can assume that  $\mathbb{P}(Z = 0) < 1$ . Since  $Z$  has mean 0,

$$\mathbb{E}|Z| = 2\mathbb{E}Z\mathbf{1}_{Z \geq 0} \leq 2(\mathbb{E}Z^4)^{1/4} \mathbb{P}(Z \geq 0)^{3/4}.$$

Moreover, by Hölder's inequality,  $\mathbb{E}|Z| \geq \frac{(\mathbb{E}Z^2)^{3/2}}{(\mathbb{E}Z^4)^{1/2}}$ , so

$$\mathbb{P}(Z \geq 0) \geq 16^{-1/3} \frac{(\mathbb{E}Z^2)^2}{\mathbb{E}Z^4}.$$

Using independence,  $\mathbb{E}Z_i = 0$  and the assumption  $\mathbb{E}Z_i^4 \leq C(\mathbb{E}Z_i^2)^2$ , we have

$$\begin{aligned}\mathbb{E}Z^4 &= \sum_{i=1}^n \mathbb{E}Z_i^4 + 6 \sum_{i<j} \mathbb{E}Z_i^2 \mathbb{E}Z_j^2 \leq \max\{C, 3\} \left( \sum_{i=1}^n (\mathbb{E}Z_i^2)^2 + 2 \sum_{i<j} \mathbb{E}Z_i^2 \mathbb{E}Z_j^2 \right) \\ &= \max\{C, 3\} (\mathbb{E}Z^2)^2.\end{aligned}$$

□

Take  $Z_i = a_i(Y_i - 1)$ . We have,  $\mathbb{E}(Y_i - 1)^2 = 1$ ,  $\mathbb{E}(X_i - \gamma)^4 = 9$ . Thus we can apply Lemma 3 with  $C = 9$  and obtain

$$(4) \quad \mathbb{P}\left(\sum a_i^2 Y_i \geq \sum a_i^2\right) \geq \frac{1}{9 \cdot 16^{1/3}}.$$

By (3),

$$\mathbb{P}\left(G \geq \sqrt{\theta^{-1}t}\right) = \mathbb{P}(G \geq t) \geq \frac{1}{2\sqrt{2\pi}} \frac{1}{t} e^{-t^2/2} \geq \frac{1}{2\sqrt{2\pi}} \frac{1}{\sqrt{\alpha t}} e^{-\alpha t/2},$$

where in the last inequality we use that in this case  $t \leq \sqrt{\alpha t}$ . Moreover, since  $\alpha t \geq \sqrt{2}$ ,  $e^{-\alpha t/2} \geq e^{1/\sqrt{2}} e^{-\alpha t}$ . Thus,

$$\mathbb{P}(S \geq t\sigma) \geq \frac{e^{1/\sqrt{2}}}{18 \cdot 16^{1/3} \sqrt{2\pi}} \frac{1}{\sqrt{\alpha t}} \exp(-\alpha t) > \frac{1}{57} \frac{1}{\sqrt{\alpha t}} \exp(-\alpha t).$$

Combining Case 1 and 2 finishes the proof of the lower bound in (2) and thus the proof of Theorem 1 is complete.  $\square$

### 3. GENERALISATIONS

In this section, we provide general tail bounds for weighted sums of nonnegative random variables which for certain distributions allow to capture the same behaviour as featured in Janson's inequality (1), viz. asymptotically the sum has the same tail as the summand carrying the largest weight.

**Theorem 4.** *Let  $X_1, \dots, X_n$  be i.i.d. nonnegative random variables,  $\mu = \mathbb{E}X_1$ . Let  $S = \sum_{i=1}^n a_i X_i$  with  $a_1, \dots, a_n$  positive. For every  $t > 1$ ,*

$$(5) \quad \mathbb{P}(S \geq \mathbb{E}S) r((t-1)\alpha\mu) \leq \mathbb{P}(S > t\mathbb{E}S) \leq \exp\{-\alpha I(\mu t)\},$$

where  $\alpha = \frac{\sum_{i=1}^n a_i}{\max_{i \leq n} a_i}$ , for  $v > 0$ ,

$$(6) \quad r(v) = \inf_{u>0} \frac{\mathbb{P}(X_1 > u+v)}{\mathbb{P}(X_1 > u)}$$

and for  $t > 0$ ,

$$(7) \quad I(t) = \sup_{\theta>0} (t\theta - \log \mathbb{E}e^{\theta X_1}).$$

Before presenting the proof, we look at the example of the exponential and gamma distribution.

**3.1. Examples.** When the  $X_i$  are exponential rate 1 random variables,  $I(t) = t - 1 - \log t$ ,  $r(v) = e^{-v}$ ,  $\mathbb{P}(S \geq \mathbb{E}S) \geq \frac{1}{9 \cdot 16^{1/3}}$  (see (4)) and we obtain

$$\frac{1}{9 \cdot 16^{1/3}} e^{-\alpha(t-1)} \leq \mathbb{P}(S > t\mathbb{E}S) \leq e^{-\alpha(t-1-\log t)}.$$

Comparing with (1), the extra factor  $\frac{1}{t}$  in the upper bound was obtained in [11] through rather delicate computations for the moment generating function specific for the exponential distribution. Since  $\alpha \geq 1$ , our lower bound up to a universal constant recovers the one from (1) (improves on it as long as  $\alpha > 9 \cdot 16^{1/3}/(2e)$  and is worse otherwise). Along the same lines, for the gamma distribution with parameter  $\gamma > 0$  (i.e. with density  $\Gamma(\gamma)^{-1} x^{\gamma-1} e^{-x}$ ,  $x > 0$ ), we have  $\mu = \gamma$ ,  $I(t\mu) = \gamma(t-1-\log t)$  and with some extra work,

$$r_\gamma(v) = \begin{cases} \frac{1}{2\Gamma(\gamma)} \min\{v^{\gamma-1}, 1\} e^{-v}, & 0 < \gamma < 1, \\ e^{-v}, & \gamma \geq 1. \end{cases}$$

Moreover, via Lemma 3,  $\mathbb{P}(S \geq \mathbb{E}S) > \frac{1}{3 \cdot 16^{1/3}(1+2\gamma^{-1})}$ . Then (5) yields

$$(8) \quad \frac{1}{3 \cdot 16^{1/3}(1+2\gamma^{-1})} r_\gamma(\alpha\gamma(t-1)) \leq \mathbb{P}(S > t\mathbb{E}S) \leq \exp(-\alpha\gamma(t-1-\log t)).$$

In particular,  $\mathbb{P}(S > t\mathbb{E}S) = \exp\{-\alpha\gamma t + o(t)\}$  as  $t \rightarrow \infty$ . It would perhaps be interesting to find a larger class of distributions for which the upper and lower bounds from (5) are asymptotically tight. For more precise results involving the variance of  $S$  for weighted sums of independent Gamma random variables (not necessarily with the same parameter), we refer to Theorem 2.57 in [1].

**3.2. Proof of Theorem 4: the upper bound.** For the log-moment generating function  $\psi: \mathbb{R} \rightarrow (-\infty, \infty]$ ,

$$\psi(u) = \log \mathbb{E}e^{uX_1}, \quad u \in \mathbb{R},$$

we have  $\psi(0) = 0$ ,  $\psi$  is convex (by Hölder's inequality). Thus, by the monotonicity of slopes of convex functions,

$$(9) \quad \mathbb{R} \ni u \mapsto \frac{\psi(u)}{u} \text{ is nondecreasing.}$$

This is what Janson's proof specified to the case of exponentials relies on. We turn to estimating the tails (using of course Chernoff-type bounds). Fix  $t > 1$ . For  $\theta > 0$ , we have

$$\begin{aligned} \mathbb{P}(S \geq t\mathbb{E}S) &= \mathbb{P}(e^{\theta S} \geq e^{\theta t\mathbb{E}S}) \leq e^{-\theta t\mathbb{E}S} \mathbb{E}e^{\theta S} = e^{-\theta t\mathbb{E}S} \prod_{i=1}^n \mathbb{E}e^{\theta a_i X_i} \\ &= \exp\left\{-\theta t\mathbb{E}S + \sum_{i=1}^n \psi(\theta a_i)\right\}. \end{aligned}$$

Let  $a_* = \max_{i \leq n} a_i$ . Thanks to (9),

$$\sum_{i=1}^n \psi(\theta a_i) = \sum_{i=1}^n (\theta a_i) \frac{\psi(\theta a_i)}{\theta a_i} \leq \sum_{i=1}^n (\theta a_i) \frac{\psi(\theta a_*)}{\theta a_*} = \frac{\sum_{i=1}^n a_i}{a_*} \psi(\theta a_*) = \alpha \psi(\theta a_*),$$

where we set  $\alpha = \frac{\sum_{i=1}^n a_i}{a_*}$ . Note  $\mathbb{E}S = \mu \sum a_i = \mu \alpha a_*$ . We obtain,

$$\mathbb{P}(S \geq t\mathbb{E}S) \leq \exp\{-\theta t\mathbb{E}S + \alpha \psi(\theta a_*)\} = \exp\{-\alpha(t\mu\theta a_* - \psi(\theta a_*))\},$$

so optimising over  $\theta$  gives the upper bound of (5).  $\square$

**3.3. Proof of Theorem 4: the lower bound.** We follow a general idea from [11]. The whole argument is based on the following simple lemma.

**Lemma 5.** *Suppose  $X$  and  $Y$  are independent random variables and  $Y$  is such that  $\mathbb{P}(Y \geq u+v) \geq r(v)\mathbb{P}(Y \geq u)$  for all  $u \in \mathbb{R}$  and  $v > 0$ , for some function  $r(v)$ . Then  $\mathbb{P}(X+Y \geq u+v) \geq r(v)\mathbb{P}(X+Y \geq u)$  for all  $u \in \mathbb{R}$  and  $v > 0$ .*

*Proof.* By independence, conditioning on  $X$ , we get

$$\begin{aligned} \mathbb{P}(X+Y \geq u+v) &= \mathbb{E}_X \mathbb{P}_Y(Y \geq u-X+v) \geq r(v) \mathbb{E}_X \mathbb{P}_Y(Y \geq u-X) \\ &= r(v) \mathbb{P}(X+Y \geq u). \end{aligned}$$

$\square$

Let  $S = \sum_{i=1}^n a_i X_i$  be the weighted sum of i.i.d. random variables and without loss of generality let us assume  $a_1 = \max_{i \leq n} a_i$ . Fix  $t > 1$ . We write  $S = S' + a_1 X_1$ , with  $S' = \sum_{i=2}^n a_i X_i$ . Note that the definition of function  $r$  from (6) remains unchanged if the infimum is taken over all  $u \in \mathbb{R}$  (since  $X_1$  is nonnegative). Thus Lemma 5 gives

$$\mathbb{P}(S \geq t\mathbb{E}S) = \mathbb{P}(S \geq \mathbb{E}S + (t-1)\mathbb{E}S) \geq r\left((t-1)\frac{\mathbb{E}S}{a_1}\right) \mathbb{P}(S \geq \mathbb{E}S),$$

as desired.  $\square$

#### 4. FURTHER REMARKS

**4.1. Moments.** The upper bound from (2) allows us to recover precise estimates for moments (a special case of Gluskin and Kwapien results from [8]), with a straightforward proof. Here and throughout,  $\|a\|_p = (\sum_{i=1}^n |a_i|^p)^{1/p}$  denotes the  $p$ -norm of a sequence  $a = (a_1, \dots, a_n)$ ,  $p > 0$ , and  $\|a\|_\infty = \max_{i \leq n} |a_i|$ .

**Theorem 6** (Gluskin and Kwapien, [8]). *Under the assumptions of Theorem 1, for every  $p \geq 2$ ,*

$$(10) \quad \frac{\sqrt{2e}}{\sqrt{2e+1}}(p\|a\|_\infty + \sqrt{p}\|a\|_2) \leq (\mathbb{E}|S|^p)^{1/p} \leq 4\sqrt{2}(p\|a\|_\infty + \sqrt{p}\|a\|_2).$$

*Proof.* For the upper bound, letting  $\tilde{S} = \frac{S}{\sqrt{\text{Var}(S)}}$  and using (2), we get

$$\mathbb{E}|\tilde{S}|^p = \int_0^\infty pt^{p-1} \mathbb{P}(|\tilde{S}| > t) dt \leq \int_0^1 pt^{p-1} dt + 2 \int_1^\infty pt^{p-1} \exp\left(-\frac{\alpha^2}{2} h\left(\frac{2t}{\alpha}\right)\right) dt.$$

We check that as  $u$  increases,  $h(u)$  behaves first quadratically, then linearly. More precisely,

$$(11) \quad h(u) \geq \frac{1}{5}u^2, \quad u \in (0, \sqrt{2}), \quad h(u) \geq \frac{1}{4}u, \quad u \in (\sqrt{2}, \infty).$$

Thus the second integral  $\int_1^\infty \dots dt$  can be upper bounded by (recall that  $\text{Var}(S) = 2\|a\|_2^2$ ,  $\frac{\alpha}{\sqrt{2}} = \frac{\|a\|_2}{\|a\|_\infty} > 1$ ),

$$\begin{aligned} & \int_1^{\alpha/\sqrt{2}} pt^{p-1} \exp\left(-\frac{\alpha^2}{2} \frac{1}{5} \left(\frac{2t}{\alpha}\right)^2\right) dt + \int_{\alpha/\sqrt{2}}^\infty pt^{p-1} \exp\left(-\frac{\alpha^2}{2} \frac{1}{4} \frac{2t}{\alpha}\right) dt \\ & \leq \int_0^\infty pt^{p-1} \exp\left(-\frac{2}{5}t^2\right) dt + \int_0^\infty pt^{p-1} \exp\left(-\frac{1}{4}\alpha t\right) dt \\ & = \left(\frac{5}{2}\right)^{p/2} \Gamma\left(\frac{p}{2} + 1\right) + \left(\frac{4}{\alpha}\right)^p \Gamma(p+1). \end{aligned}$$

Using  $\Gamma(x+1) \leq x^x$ ,  $x \geq 1$ , yields

$$\begin{aligned} (\mathbb{E}|S|^p)^{1/p} &= \sqrt{2}\|a\|_2 (\mathbb{E}|\tilde{S}|^p)^{1/p} \leq \sqrt{2}\|a\|_2 \left(1 + 2\left(\frac{5p}{4}\right)^{p/2} + 2\left(\frac{4p}{\alpha}\right)^p\right)^{1/p} \\ &\leq 4\sqrt{2}(p\|a\|_\infty + \sqrt{p}\|a\|_2). \end{aligned}$$

For the lower bound, suppose  $a_1 = \|a\|_\infty$ . Then, by independence and Jensen's inequality,

$$\mathbb{E}|S|^p \geq \mathbb{E}|a_1 X_1 + \mathbb{E}(a_2 X_2 + \dots + a_n X_n)|^p = a_1^p \mathbb{E}|X_1|^p = a_1^p \Gamma(p+1).$$

Using  $\Gamma(x+1)^{1/x} \geq x/e$ ,  $x > 0$  (Stirling's formula, [10]), this gives

$$(\mathbb{E}|S|^p)^{1/p} \geq \frac{p}{e} \|a\|_\infty.$$

On the other hand, by Proposition 2, and Jensen's inequality,

$$\mathbb{E}|S|^p = \mathbb{E} \left( 2 \sum a_i^2 Y_i \right)^{p/2} \mathbb{E}|G|^p \geq \left( 2 \sum a_i^2 \right)^{p/2} \mathbb{E}|G|^p.$$

Using  $\mathbb{E}|G|^p \geq (p/e)^{p/2}$ ,  $p \geq 1$  (again, by e.g. Stirling's approximation), we obtain

$$(\mathbb{E}|S|^p)^{1/p} \geq \sqrt{\frac{2}{e}} \sqrt{p} \|a\|_2.$$

Combining gives

$$(\mathbb{E}|S|^p)^{1/p} \geq \max \left\{ \frac{1}{e} p \|a\|_\infty, \sqrt{\frac{2}{e}} \sqrt{p} \|a\|_2 \right\} \geq \frac{\sqrt{2e}}{\sqrt{2e+1}} (p \|a\|_\infty + \sqrt{p} \|a\|_2),$$

which finishes the proof.  $\square$

*Remark 7.* Using Markov and Payley-Zygmund type inequalities, it is possible to recover two-sided tail bounds from moment estimates (like (10)), but incurring loss of (universal) constants in the exponents, as it is done in e.g. [8], or [9].

**4.2. Upper bounds on upper tails from S-inequalities.** Let  $S$  be as in (1). The upper bound in (1) for  $t = 1$  is trivial, whereas as a result of Lemma 3, viz. (4), we obtain  $\mathbb{P}(S \geq \mathbb{E}S) \in (\frac{1}{24}, \frac{23}{24})$ , where the upper bound  $\frac{23}{24}$  is obtained by applying Lemma 3 to  $-Z$ . Letting  $a > 0$  be such that  $\mathbb{P}(S \geq \mathbb{E}S) = \mathbb{P}(X_1 \geq a) = e^{-a}$ , by the S-inequality for the two-sided product exponential measure and the set  $\{x \in \mathbb{R}^n, \sum a_i |x_i| \leq \mathbb{E}S\}$  (Theorem 2 in [13]), we obtain that for every  $t \geq 1$ ,

$$(12) \quad \mathbb{P}(S \geq t\mathbb{E}S) \leq \mathbb{P}(X_1 \geq ta) = e^{-at} \leq \left(\frac{23}{24}\right)^t.$$

This provides an improvement of (1) for small enough  $t$  (of course the point of (1) is that it is optimal for large  $t$ ). The same can be said about the upper bound in (8) for  $\gamma \geq 1$  (in view of (4) and the results from [14] for gamma distributions with parameter  $\gamma \geq 1$ ). Complimentary to such concentration bounds are small ball probability estimates and anti-concentration phenomena, typically treating however the regime of  $t = O(1/\mathbb{E}S)$  (under our normalisation). We refer for instance to the comprehensive survey [16] of Nguyen and Vu, as well as the recent work [12] of Li and Madiman for further results and references. Specific reversals of (12) concerning the exponential measure can be found e.g. in [5] (Corollary 15), [18] (Proposition 3.4), [19] ((5.5) and Theorem 5.7).

**4.3. Heavy-tailed distributions.** Janson's as well as this paper's techniques strongly rely on Chernoff-type bounds involving exponential moments to establish the *largest-weight* summand tail asymptotics from (1) or (2). Interestingly, when the exponential moments do *not* exist, i.e. for heavy-tailed distributions, under some natural additional assumptions (subexponential distributions), a different phenomenon occurs: in the simplest case of i.i.d. summands, we have

$$\mathbb{P}(X_1 + \dots + X_n > t) = (1 + o(1)) \mathbb{P} \left( \max_{i \leq n} X_i > t \right) \quad \text{as } t \rightarrow \infty,$$

often called the single big jump or catastrophe principle. We refer to the monograph [7] (Chapters 3.1 and 5.1), as well as the papers [3] and [6] for extensions including weighted sums and continuous time respectively.

**4.4. Theorem 1 in a more general framework.** A careful inspection of the proof of Theorem 1 shows that thanks to Theorem 2.57 from [1] (or the simpler but weaker bound (8)), the former can be extended to the case where the  $X_i$  have the same distribution as  $\sqrt{Y_i}G_i$  with the  $Y_i$  being i.i.d. gamma random variables and the  $G_i$  independent standard Gaussian. For simplicity, we have decided to present it for the symmetric exponentials.

## REFERENCES

- [1] Bercu, B., Delyon, B., Rio, E., Concentration inequalities for sums and martingales. SpringerBriefs in Mathematics. Springer, Cham, 2015.
- [2] Boucheron, S., Lugosi, G., Massart, P., Concentration inequalities. A nonasymptotic theory of independence. With a foreword by Michel Ledoux. Oxford University Press, Oxford, 2013.
- [3] Chen, Y., Ng, K. W., Tang, Q., Weighted sums of subexponential random variables and their maxima. Adv. in Appl. Probab. 37 (2005), no. 2, 510–522.
- [4] Eskenazis, A. On Extremal Sections of Subspaces of  $L_p$ . Discrete Comput. Geom. (2019).
- [5] Eskenazis, A., Nayar, P., Tkocz, T., Gaussian mixtures: entropy and geometric inequalities, Ann. Probab. 46 (2018), no. 5, 2908–2945.
- [6] Foss, S., Konstantopoulos, T., Zachary, S., Discrete and continuous time modulated random walks with heavy-tailed increments. J. Theoret. Probab. 20 (2007), no. 3, 581–612.
- [7] Foss, S., Korshunov, D., Zachary, S., An introduction to heavy-tailed and subexponential distributions. Second edition. Springer Series in Operations Research and Financial Engineering. Springer, New York, 2013.
- [8] Gluskin, E. D., Kwapien, S., Tail and moment estimates for sums of independent random variables with logarithmically concave tails. Studia Math. 114 (1995), no. 3, 303–309.
- [9] Hitczenko, P., Montgomery-Smith, S., A note on sums of independent random variables. Advances in stochastic inequalities (Atlanta, GA, 1997), 69–73,
- [10] Jameson, G., A simple proof of Stirling’s formula for the gamma function. Math. Gaz. 99 (2015), no. 544, 68–74.
- [11] Janson, S., Tail bounds for sums of geometric and exponential variables. Statist. Probab. Lett. 135 (2018), 1–6.
- [12] Li, J., Madiman, M., A combinatorial approach to small ball inequalities for sums and differences. Combin. Probab. Comput. 28 (2019), no. 1, 100–129.
- [13] Nayar, P., Tkocz, T., The unconditional case of the complex S-inequality. Israel J. Math. 197 (2013), no. 1, 99–106.
- [14] Nayar, P., Tkocz, T., S-inequality for certain product measures. Math. Nachr. 287 (2014), no. 4, 398–404.
- [15] Nayar, P., Tkocz, T., On a convexity property of sections of the cross-polytope. Proc. Amer. Math. Soc. 148 (2020), no. 3, 1271–1278.
- [16] Nguyen, H. H., Vu, V. H., Small ball probability, inverse theorems, and applications. Erdős centennial, 409–463, Bolyai Soc. Math. Stud., 25, János Bolyai Math. Soc., Budapest, 2013.
- [17] Oleszkiewicz, K., Precise moment and tail bounds for Rademacher sums in terms of weak parameters. Israel J. Math. 203 (2014), no. 1, 429–443.
- [18] Paouris, G., Valettas, P., A Gaussian small deviation inequality for convex functions. Ann. Probab. 46 (2018), no. 3, 1441–1454.
- [19] Paouris, G., Valettas, P., Variance estimates and almost Euclidean structure. Adv. Geom. 19 (2019), no. 2, 165–189.

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