A note on certain convolution operators

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Abstract

In this note we consider a certain class of convolution operators acting on the L_p spaces of the one dimensional torus. We prove that the identity minus such an operator is nicely invertible on the subspace of functions with mean zero.

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1 Introduction

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the one dimensional torus viewed as a compact group with the addition modulo 1, $x \oplus y = (x+y) \mod 1$, $x, y \in \mathbb{R}$ equipped with the Haar measure — the unique invariant probability measure (the Lebesgue measure). To begin with, fix $1 \leq p \leq \infty$ and consider the averaging operator U_t acting on $L_p(\mathbb{T})$ (with the usual norm $\|f\| = \left(\int_{\mathbb{T}} |f|^p\right)^{1/p}$ for $p < \infty$, and $\|f\| = \operatorname{ess\,sup}_{\mathbb{T}} |f|$ for $p = \infty$)

$$(U_t f)(x) = \frac{1}{2t} \int_{-t}^{+t} f(x \oplus s) \, \mathrm{d}s, \quad t \in (0,1). \tag{1}$$

If t is small, is the operator $I - U_t$ invertible, or, in other words, how much does $U_t f$ differ from f? Of course, averaging a constant function does not change it, but excluding such a trivial case, we get a quantitative answer.

Theorem 1. *Let $t \in (0,1)$. There exists a universal constant c such that for every $1 \leq p \leq \infty$ and every $f \in L_p(\mathbb{T})$ with $\int_{\mathbb{T}} f = 0$ we have*

$$\|f - U_t f\| \geq ct^2 \|f\|, \tag{2}$$

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where $\|\cdot\|$ *denotes the* L_p *norm.*

Note that if p was equal to 2, then, with the aid of the Fourier analysis, the above estimate would be trivial. However, $\|\cdot\|$ is set to be the L_p norm for some $1 \leq p \leq \infty$, the constant does not depend on p, therefore the situation is more subtle.

When $p = 1$, if we further estimate the left hand side of (2) using the Sobolev inequality, see [GT], we obtain the following corollary.

Corollary 1. *Let us consider* $t \in (0, 1)$ *and assume that* f *belongs to the Sobolev space* $W^{1,1}(\mathbb{T})$ *with* $\int_{\mathbb{T}} f = 0$. *Then we have*

$$\int_{\mathbb{T}} \left| f'(x) - \frac{f(x \oplus t) - f(x \oplus -t)}{2t} \right| dx \geq ct^2 \int_{\mathbb{T}} |f(x)| \, dx, \tag{3}$$

where $c > 0$ *is a universal constant.*

Remark. Setting $t = 1/2$, inequality (3) becomes the usual Sobolev inequality, so (3) can be viewed as a certain generalization of the Sobolev inequality.

Remark. Set $f(x) = \cos(2\pi x)$. Then $\|f - U_t f\| = \|f\| \left(1 - \frac{1}{2\pi t} \sin(2\pi t)\right) \approx t^2 \|f\|$, for small t. Therefore, the inequality in Theorem 1 is sharp in a sense.

In this note we give a proof of a generalization of Theorem 1. We say that a $\mathbb{T}$-valued random variable Z is *c-good* with some positive constant c if $\mathbb{P}(Z \in A) \geq c|A|$ for all measurable $A \subset \mathbb{T}$. Equivalently, by Lebesgue's decomposition theorem it means that the absolutely continuous part of Z (with respect to the Lebesgue measure) has a density bounded below by a positive constant. We say that a real random variable Y is *ℓ-decent* if $Y_1 + \ldots + Y_\ell$ has a nontrivial absolutely continuous part, where $Y_1, Y_2, \ldots$ are i.i.d. copies of Y. Our main result reads

Theorem 2. *Given* $t \in (0, 1)$ *and an* ℓ*-decent real random variable* Y*, consider the operator* A_t *given by*

$$(A_t f)(x) = \mathbb{E} f(x \oplus tY). \tag{4}$$

Then there exists a positive constant c *which depends only on the distribution of the random variable* Y *such that for every* $1 \leq p \leq \infty$ *and every* $f \in L_p(\mathbb{T})$ *with* $\int_{\mathbb{T}} f = 0$ *we have*

$$\|f - A_t f\| \geq ct^2 \|f\|,$$

where $\|\cdot\|$ *denotes the* L_p *norm.*

Remark. One cannot hope to prove a statement similar to Theorem 2 for purely atomic measures. Indeed, just consider the case $p = 1$ and let Y be distributed according to the law $\mu_Y = \sum_{i=1}^{\infty} p_i \delta_{x_i}$. Then for every $\varepsilon > 0$ and every $t \in (0, 1)$ there exists $f \in L_1(\mathbb{T})$ such that $\|f - A_t(f)\| < \varepsilon$ and $\|f\| = 1$. To see this take N such that $\sum_{i=N+1}^{\infty} p_i < \varepsilon/4$ and let $f_n(x) = \frac{\pi}{2}\sin(2\pi n x)$. Then $\|f_n\| = 1$. Let $n_0 \geq 8\pi/\varepsilon$. Consider a sequence $\big((\pi n t x_1 \mod 2\pi, \ldots, \pi n t x_N \mod 2\pi)\big)_n$ for $n = 0, 1, 2, \ldots, n_0^N$ and observe that by the pigeonhole principle there exist $0 \leq n_1 < n_2 \leq n_0^N$ such that for all $1 \leq i \leq N$ we have $\operatorname{dist}(\pi t x_i(n_1 - n_2), 2\pi\mathbb{Z}) \leq \frac{2\pi}{n_0}$. Taking $n = n_2 - n_1$ we obtain

$$\begin{aligned}
\|f_n - A_t(f_n)\| &\leq \frac{\pi}{2}\sum_{i=1}^{N} p_i \,\|\sin(2\pi n x) - \sin(2\pi n(x + t x_i))\| + \frac{\varepsilon}{2} \\
&= \pi \sum_{i=1}^{N} p_i |\sin(\pi n t x_i)| \cdot \|\cos(2\pi n x \oplus \pi n t x_i)\| + \frac{\varepsilon}{2} \\
&\leq 2\sum_{i=1}^{N} p_i |\sin(\pi n t x_i)| + \frac{\varepsilon}{2} \leq \frac{4\pi}{n_0}\sum_{i=1}^{N} p_i + \frac{\varepsilon}{2} \leq \varepsilon. \quad \square
\end{aligned}$$

Our result gives the bound for the norm of an operator of the form $(I - A_t)^{-1}$. The main difficulty is that this operator is not globally invertible. Of course, boundedness of a resolvent operator $R_\lambda(A) = (A - zI)^{-1}$ has been thoroughly studied (see e.g. [G], [Ba] which feature Hilbert space setting for Hilbert-Schmidt and Schatten-von Neumann operators). Let us also mention that the first part of the book [CE] is a set of related articles concerning mainly the problem of finding the inverse formula for certain Toeplitz-type operators. The paper [GS] contains the famous Gohberg-Semencul formula for the inverse of a non-Hermitian Toeplitz matrix. In [GH] the authors generalized the results of [CE] to the case of Toeplitz matrices whose entries are taken from some noncommutative algebra with a unit. The operators of the form $I - K$ (acting e.g. on $L_1([0, 1])$), where K is a certain operator with a kernel $k(t - s)$, are continuous versions of the operators given by Toeplitz matrices. Paper [GS] deals also with this kind of operators, namely

$$(I - K)(f)(x) = f(x) - \int_0^1 k(t - s) f(s)\, \mathrm{d}s,$$

where $k \in L_1([-1, 1])$. In the case of $I - K$ being invertible, the authors give a formula for the inverse operator $(I - K)^{-1}$ in terms of solutions of certain four integral equations. See also Article 3 in [CE] for generalizations of these formulas.

2 Proof of Theorem 2

We begin with two lemmas.

Lemma 1. *Suppose Y is an ℓ-decent random variable. Let $Y_1, Y_2, \ldots$ be independent copies of Y. Then there exist a positive integer $N = N(Y)$ and numbers $c = c(Y) > 0$, $C_0 = C_0(Y) \geq 1$ such that for all $C \geq C_0$ and $n \geq N$ the random variable*

$$X_n^{(C)} = \left(C \cdot \frac{Y_1 + \ldots + Y_n}{\sqrt{n}}\right) \mod 1 \tag{5}$$

is c-good.

Proof. We prove the lemma in a few steps considering more and more general assumptions about Y.

Step I. Suppose that the characteristic function of Y belongs to $L_p(\mathbb{R})$ for some $p \geq 1$. In this case, by a certain version of the Local Central Limit Theorem, e.g. Theorem 19.1 in [BR], p. 189, we know that the density q_n of $(Y_1 + \ldots + Y_n - n\mathbb{E}Y)/\sqrt{n}$ exists for sufficiently large n, and satisfies

$$\sup_{x \in \mathbb{R}} \left| q_n(x) - \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2} \right| \xrightarrow[n \to \infty]{} 0, \tag{6}$$

where $\sigma^2 = \mathrm{Var}(Y)$. Observe that the density $g_n^{(C)}$ of $X_n^{(C)}$ equals

$$g_n^{(C)}(x) = \sum_{k \in \mathbb{Z}} \frac{1}{C} q_n\left(\frac{1}{C}(x+k) - \sqrt{n}\mathbb{E}Y\right), \qquad x \in [0,1].$$

Using (6), for $\delta = \frac{e^{-2/\sigma^2}}{\sqrt{2\pi}\sigma}$ we can find $N = N(Y)$ such that

$$q_n(x) > \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2} - \delta/8, \qquad x \in \mathbb{R},\ n \geq N.$$

Therefore, to be close to the maximum of the Gaussian density we sum over only those k's for which $x + k \in (-2C, 2C) + C\sqrt{n}\mathbb{E}Y$ for all $x \in [0,1]$. Since there are at least C and at most $4C$ such k's, we get that

$$g_n^{(C)}(x) > \frac{1}{C} \frac{1}{\sqrt{2\pi}\sigma} e^{-2/\sigma^2} \cdot C - \frac{1}{C} \frac{\delta}{8} \cdot 4C = \frac{1}{2\sqrt{2\pi}\sigma} e^{-2/\sigma^2}.$$

In particular, it implies that $X_n^{(C)}$ is c-good with $c = \frac{1}{2\sqrt{2\pi}\sigma} e^{-2/\sigma^2}$. Thus, in this case, it suffices to set $C_0 = 1$.

Step II. Suppose that the law of Y is of the form $q\mu + (1-q)\nu$ for some $q \in (0,1]$ and some Borel probability measures μ, ν on $\mathbb{R}$ such that the characteristic function of μ belongs to $L_p(\mathbb{R})$ for some $p \geq 1$. Notice that

$$\mu_{Y_1+\ldots+Y_N} = \mu_Y^{\star N} = (q\mu + (1-q)\nu)^{\star N} = \sum_{k=0}^{N} \binom{N}{k} q^k (1-q)^{N-k} \mu^{\star k} \star \nu^{\star (N-k)}$$

$$\geq \sum_{k=N_0}^{N} \binom{N}{k} q^k (1-q)^{N-k} \mu^{\star k} \star \nu^{\star (N-k)} = c_{N,N_0} \left(\mu^{\star N_0} \star \rho_{N,N_0} \right),$$

where

$$\rho_{N,N_0} = \frac{1}{c_{N,N_0}} \sum_{k=N_0}^{N} \binom{N}{k} q^k (1-q)^{N-k} \mu^{\star k - N_0} \star \nu^{\star (N-k)}$$

is a probability measure, and

$$c_{N,N_0} = \sum_{k=N_0}^{N} \binom{N}{k} q^k (1-q)^{N-k}$$

is a normalisation constant. Choosing $N_0 = \lfloor qN - C_1\sqrt{q(1-q)N} \rfloor$ we can guarantee that $c_{N,N_0} \geq 1/2$ eventually, say for $N \geq \tilde{N}$. Denoting by $\bar{Y}$, Z the random variables with the law μ, ρ_{N,N_0} respectively and by $\bar{Y}_i$ i.i.d. copies of $\bar{Y}$, we get

$$\mathbb{P}\left(X_N^{(C)} \in A\right) \geq c_{N,N_0} \mathbb{P}\left(\left(C\frac{\bar{Y}_1 + \ldots + \bar{Y}_{N_0}}{\sqrt{N}} + C\frac{Z_{N,N_0}}{\sqrt{N}} \right) \mod 1 \in A \right).$$

By Step I, the first bit $C(\bar{Y}_1 + \ldots + \bar{Y}_{N_0})/\sqrt{N}$ is c-good for some $c > 0$ and $C \geq C_0^{(II)} = \sup_{N \geq \tilde{N}} \sqrt{N/N_0}$. Moreover, note that if U is a c-good $\mathbb{T}$-valued r.v., then so is $U \oplus V$ for every $\mathbb{T}$-valued r.v. V which is independent of U. As a result, $X_N^{(C)}$ is $c/2$-good.

Step III. Now we consider the general case, i.e. Y is ℓ-decent for some $\ell \geq 1$. For $n \geq \ell$ we can write

$$C \cdot \frac{Y_1 + \ldots + Y_n}{\sqrt{n}} = C\sqrt{\frac{\lfloor n/\ell \rfloor}{n}} \cdot \frac{\tilde{Y}_1 + \ldots + \tilde{Y}_{\lfloor n/\ell \rfloor}}{\sqrt{\lfloor n/\ell \rfloor}} + C\frac{\tilde{R}}{\sqrt{n}}$$

with $\tilde{Y}_j = Y_{(j-1)\ell+1} + \ldots + Y_{j\ell}$ for $j = 1, \ldots, \lfloor n/\ell \rfloor$, and $\tilde{R} = Y_{\lfloor n/\ell \rfloor \ell + 1} + \ldots + Y_n$. Since the absolutely continuous part of the law μ of $\tilde{Y}_j$ is nontrivial, then

μ is of the form $q\nu_1+(1-q)\nu_2$ with $q \in (0,1]$ and the characteristic function of ν_1 belonging to some L_p. Indeed, μ has a bit which is a uniform distribution on some measurable set whose characteristic function is in L_2. Therefore, applying Step II for $\tilde{Y}_j$'s we get that $X_n^{(C)}$ is c-good when $C\sqrt{\frac{\lfloor n/\ell \rfloor}{n}} \geq C_0^{(II)}$. So we can set $C_0 = C_0^{(II)}\sqrt{2\ell}$. □

Lemma 2. *Suppose Z is a $\mathbb{T}$-valued c-good random variable and B_Z is the operator defined by $(B_Z f)(x) = \mathbb{E}f(x \oplus Z)$. Then for every $1 \leq p \leq \infty$ and every $f \in L_p(\mathbb{T})$ with $\int_{\mathbb{T}} f = 0$ we have $\|B_Z f\| \leq (1-c)\|f\|$, where $\|\cdot\|$ is the L_p norm.*

Proof. Fix $1 \leq p < \infty$. Let μ be the law of Z. Define the measure $\nu(A) = (\mu(A) - c|A|)/(1-c)$ for measurable $A \subset \mathbb{T}$. Since μ is c-good, ν is a Borel probability measure on $\mathbb{T}$. Take $f \in L_p(\mathbb{T})$ with mean zero. Then by Jensen's inequality we have

$$\begin{aligned}
\|B_Z f\|^p &= \int_0^1 \left| \int_0^1 f(x \oplus s)\, \mathrm{d}\mu(s) \right|^p \mathrm{d}x \\
&= (1-c)^p \int_0^1 \left| \int_0^1 f(x \oplus s)\, \mathrm{d}\nu(s) \right|^p \mathrm{d}x \\
&\leq (1-c)^p \int_0^1 \int_0^1 |f(x \oplus s)|^p\, \mathrm{d}\nu(s)\, \mathrm{d}x \\
&= (1-c)^p \|f\|^p \int_0^1 \mathrm{d}\nu(s) = (1-c)^p \|f\|^p .
\end{aligned}$$

Since c does not depend on p we get the same inequality for $p = \infty$ by passing to the limit. □

Now we are ready to give the proof of Theorem 2.

Proof of Theorem 2. Fix $1 \leq p \leq \infty$. Let $Y_1, Y_2, \ldots$ be independent copies of Y. Observe that

$$\begin{aligned}
(A_t^n f)(x) &= \mathbb{E}f\left(x \oplus tY_1 \oplus \ldots \oplus tY_n\right) \\
&= \mathbb{E}f\left(x \oplus \left(t\sqrt{n}\left(\frac{Y_1 + \ldots + Y_n}{\sqrt{n}}\right) \bmod 1\right)\right).
\end{aligned}$$

Take $n(t) = C_0^2 \lceil 1/t^2 \rceil N$, where C_0 and N are the numbers given by Lemma 1. Therefore, with $X_{n(t)}^{(C)}$ defined by (5), we can write

$$(A_t^{n(t)} f)(x) = \mathbb{E}f\left(x \oplus X_{n(t)}^{(C)}\right),$$

where $C = t\sqrt{n(t)} = tC_0\sqrt{\lceil 1/t^2\rceil N} \geq C_0\sqrt{N} \geq C_0$. Thus $X_{n(t)}^{(C)}$ is $c(Y)$-good with some constant $c(Y) \in (0,1)$. From Lemma 2 we have

$$\left\| A_t^{n(t)} f \right\| \leq (1 - c(Y)) \|f\|$$

for all f satisfying $\int_{\mathbb{T}} f = 0$.

The operator A_t is a contraction, namely $\|A_t f\| \leq \|f\|$ for all $f \in L_1(\mathbb{T})$. Using this observation and the triangle inequality we obtain

$$\begin{aligned} \|f - A_t f\| &\geq \frac{1}{n}\left(\|f - A_t f\| + \left\|A_t f - A_t^2 f\right\| + \ldots + \left\|A_t^{n-1} f - A_t^n f\right\|\right) \\ &\geq \frac{1}{n}\|f - A_t^n f\|. \end{aligned}$$

Taking $n = n(t)$ we arrive at

$$\frac{1}{n(t)}\left\|f - A_t^{n(t)} f\right\| \geq \frac{1}{t^{-2}+1} \cdot \frac{1}{C_0^2 \cdot N}\left(\|f\| - \left\|A_t^{n(t)} f\right\|\right) \geq \frac{c(Y)}{2C_0^2 \cdot N} t^2 \|f\|.$$

It suffices to take $c = c(Y)/(2C_0^2 \cdot N)$. □

Remark. Consider an ℓ-decent random variable Y. As it was noticed in the proof of Lemma 1 (Step III), the law $Y_1 + \ldots + Y_\ell$ has a bit whose characteristic function is in L_2. Conversely, if the law of $S_m = Y_1 + \ldots + Y_m$ has the form $q\mu + (1-q)\nu$ with $q \in (0,1]$ and the characteristic function of μ belonging to L_p for some $p \geq 1$, then the characteristic function of the bit $\mu^{\star\lceil p/2\rceil}$ of the sum of $\lceil p/2\rceil$ i.i.d. copies of S_m is in L_2. In particular, that bit has a density function in $L_1 \cap L_2$. Thus Y is $(m\lceil p/2\rceil)$-decent.

Remark. The idea to study the operators A_t (see (4)) stemmed from the following question posed by Gideon Schechtman (personal communication): *given $\varepsilon > 0$, is it true that there exists a natural number $k = k(\varepsilon)$ such that for any bounded linear operator $T : L_1[0,1] \to L_1[0,1]$ with $\|T\|_{L_1 \to L_1} \leq 1$ which has the property*

$$\forall f \in L_1[0,1] \ \left(|\mathrm{supp} f| \leq 1/2 \implies \|Tf\|_1 \geq \varepsilon \|f\|_1 \right)$$

there exist $\delta > 0$ and functions $g_1, \ldots, g_k \in L_\infty[0,1]$ such that

$$\|Tf\|_1 \geq \delta \|f\|_1 \quad \textit{for any } f \in L_1[0,1] \textit{ satisfying } \int_0^1 f g_j = 0,\ j = 1, \ldots, k?$$

This, in an equivalent form, was asked by Bill Johnson in relation with a question on Mathoverflow [MO]. Our hope was that an operator $T = I - A_t$, for some Y, would provide a negative answer to Schechtman's question. However, Theorem 2 says that if Y is an ℓ-decent random variable, then T is nicely invertible on the subspace of functions $f \in L_1$ such that $\int f \cdot 1 = 0$.

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[MO] `http://mathoverflow.net/questions/101253.`

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