Convexity properties of sections of 1-symmetric bodies and Rademacher sums

Joseph Kalarickal, David Rotunno, Salil Singh*, Tomasz Tkocz†

Carnegie Mellon University, Pittsburgh, PA 15213, USA

Abstract

We establish a monotonicity-type property of volume of central hyperplane sections of the 1-symmetric convex bodies, with applications to chessboard cutting. We parallel this for projections with a new convexity-type property for Rademacher sums.

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1 Introduction and results

A line can intersect at most $2N-1$ squares of the standard $N \times N$ chessboard and this is achieved by a diagonal line pushed down a bit. It is only recently that this fact has been generalised to higher dimensions and arbitrary convex bodies. Specifically, given a convex body K in $\mathbb{R}^n$ and $N \geq 1$, consider the (open) cells

*Email: salils@andrew.cmu.edu

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of the lattice $\frac{1}{N}\mathbb{Z}^n$, that is the cubes $z+(0,\frac{1}{N})^n$, $z \in \frac{1}{N}\mathbb{Z}^n$, and let $C_K(N)$ be the maximal number of cells contained in K that a hyperplane in $\mathbb{R}^n$ can intersect. For the standard cube, $K=[0,1]^n$, we simply write $C_n(N)=C_{[0,1]^n}(N)$, so $C_2(N)=2N-1$. Bárány and Frankl in [4] showed that $C_3(N)\le \frac{9}{4}N^2+2N+1$ for all $N\ge 1$ and $C_3(N)\ge \frac{9}{4}N^2+N-5$ for all N sufficiently large. In the companion work [5], they established the exact asymptotics of $C_K(N)$ as $N\to\infty$ for a fixed body K. Their main result is that

$$C_K(N)=\beta_K N^{n-1}(1+o(1)), \qquad N\to\infty,$$

with the constant β_K of the leading term given by

$$\beta_K=\max_{a\in\mathbb{R}^n\setminus\{0\}}\max_{t\in\mathbb{R}}\frac{\|a\|_1}{|a|}\operatorname{vol}_{n-1}(K\cap(ta+a^{\perp})). \tag{1}$$

Here and throughout, $|x|$ is the standard Euclidean norm, whereas $\|x\|_p$ is the ℓ_p norm of a vector x in $\mathbb{R}^n$, so $|x|=\|x\|_2$. It is a consequence of the Brunn-Minkowski inequality that when K is symmetric, say about the origin, then given an outer-normal vector a, the maximal volume section $\operatorname{vol}_{n-1}(K\cap(ta+a^{\perp}))$ is the central one at $t=0$. Thus we define the 0-homogeneous function,

$$V_K(a)=\frac{\|a\|_1}{|a|}\operatorname{vol}_{n-1}(K\cap a^{\perp}), \qquad a\in\mathbb{R}^n\setminus\{0\} \tag{2}$$

and for origin-symmetric K, we have $\beta_K=\max_a V_K(a)$.

Bárány and Frankl in [5] conjectured that for the unit cube $Q_n=[-\frac{1}{2},\frac{1}{2}]^n$, the maximum of V_{Q_n} is attained at diagonal vectors. This was confirmed by Aliev in [2],

$$\beta_{Q_n}=V_{Q_n}((1,\dots,1)).$$

We refine this result to a Schur-convexity statement (for background on majorisation, we refer for instance to Chapter II of Bhatia's book [6]). In fact, not only does this hold for the cube, but for all 1-symmetric convex bodies. A convex body K in $\mathbb{R}^n$ is called 1-symmetric if it is symmetric with respect to every coordinate hyperplane $\{x\in\mathbb{R}^n, x_j=0\}$, $j\le n$, and K is invariant under permutations of the coordinates.

Theorem 1. *Let K be a 1-symmetric convex body in $\mathbb{R}^n$. Then the function $a\mapsto V_K(a)$ defined in (2) is Schur concave on $\mathbb{R}^n_+$. In particular, for the chessboard* *cutting constant defined in* (1)*, we have* $\beta_K = \sqrt{n}\,\mathrm{vol}_{n-1}(K \cap (1,\dots,1)^\perp)$*.*

Our short proof crucially relies on Busemann's theorem from [8], combined with the symmetries of the body. In contrast, Aliev's approach from [2] employs Busemann's theorem in a further geometric argument on the plane which did not seem to allow for the present generalisation to Schur-convexity. We record Busemann's theorem for future use.

Theorem 2 (Busemann, [8])**.** *Let* K *be an origin-symmetric convex body in* $\mathbb{R}^n$*. Then the function*

$$N_K(x) = \frac{|x|}{\mathrm{vol}_{n-1}(K \cap x^\perp)}, \qquad x \neq 0 \tag{3}$$

extended at 0 *by* 0 *defines a norm on* $\mathbb{R}^n$*.*

We also refer to Theorem 3.9 in [11] for a generalisation to lower-dimensional sections, as well as to Theorem 5 in [3] for an extension to log-concave functions.

With Busemann's theorem in hand, we can motivate our next result. Hyperplane sections of the unit volume cube $Q_n = [-\frac{1}{2}, \frac{1}{2}]^n$ admit a curious probabilistic formula: if we let $\xi_1, \xi_2, \dots$ be i.i.d. random vectors uniform on the unit sphere S^2 in $\mathbb{R}^3$, then for a *unit* vector $a \in \mathbb{R}^n$, we have

$$\mathrm{vol}_{n-1}(Q_n \cap a^\perp) = \mathbb{E}\left[|a_1\xi_1 + \cdots + a_n\xi_n|^{-1}\right],$$

see [10]. Thus Busemann's theorem in particular asserts that the function

$$x \mapsto \frac{|x|}{\mathrm{vol}_{n-1}(Q_n \cap x^\perp)} = \frac{|x|}{\mathbb{E}\left[\left|\sum_{j=1}^n \frac{x_j}{|x|}\xi_j\right|^{-1}\right]} = \left(\mathbb{E}\left|\sum_{j=1}^n x_j\xi_j\right|^{-1}\right)^{-1}$$

is convex on $\mathbb{R}^n$. A perhaps much simpler (geometrically dual) analogue of this fact is that for i.i.d. Rademacher random variables $\varepsilon_1, \varepsilon_2, \dots$ (random signs, $\mathbb{P}(\varepsilon_j = \pm 1) = \frac{1}{2}$), the function

$$x \mapsto \mathbb{E}\,|x_1\varepsilon_j + \cdots + x_n\varepsilon_n|$$

is plainly convex on $\mathbb{R}^n$. Resisting great efforts and prompting significant activity across geometric functional analysis, the conjectured logarithmic Brunn-Minkowski inequality posed in [7] can be equivalently stated as a convexity property of sections of the cube (see [12]), which in particular would imply that the function

$$t \mapsto -\log\left(\mathbb{E}\left|e^{t_1}\xi_1 + \cdots + e^{t_n}\xi_n\right|^{-1}\right) \tag{4}$$

is convex on $\mathbb{R}^n$. To the best of our knowledge, even this apparent "toy-case" remains unproved. Driven by the analogy with random signs, we establish the following result.

Theorem 3. *Let $\varepsilon_1, \varepsilon_2, \ldots$ be independent Rademacher random variables. For every $n \geq 1$ and $p \geq 1$, the function*

$$\Phi(t_1, \ldots, t_n) = \log \mathbb{E}\left|e^{t_1}\varepsilon_1 + \cdots + e^{t_n}\varepsilon_n\right|^p$$

is convex on $\mathbb{R}^n$.

Our proof leverages the usual Hölder duality, but nontrivially restricted to random variables having nonnegative correlations with the random signs.

We present the proofs in the next section. The final section is devoted to further remarks. In particular, with the same method, we obtain an extension of Aliev's result from [1]. We make precise the alluded geometric duality and a connection of our Theorem 3 to Saroglou's result from [14].

2 Proofs

2.1 Proof of Theorem 1

First note that by the symmetries of K, function V_K is also symmetric (under permuting the coordinates of the input), as well as unconditional, that is $V_K(a_1, \ldots, a_n) = V_K(|a_1|, \ldots, |a_n|)$. Fix $x, y \in \mathbb{R}_+^n$ such that $x \prec y$, that is y majorises x. In particular, $\|x\|_1 = \|y\|_1$. Thus, to show $V_K(x) \geq V_K(y)$, equivalently, we would like to show that

$$\frac{1}{|x|}\operatorname{vol}_{n-1}(K \cap x^\perp) \geq \frac{1}{|y|}\operatorname{vol}_{n-1}(K \cap y^\perp),$$

that is $N(x) \leq N(y)$ with

$$N(a) = \frac{|a|}{\mathrm{vol}_{n-1}(K \cap a^{\perp})}.$$

By Theorem 2, N is convex. By the symmetries of K, function N is symmetric. Since $x \prec y$, then $y = \sum_{\sigma} \lambda_{\sigma} x_{\sigma}$ for some nonnegative weights λ_{σ} adding up to 1, where the sum is over all permutations and $x_{\sigma} = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. By the convexity of N and its symmetry,

$$N(y) = N\left(\sum \lambda_{\sigma} x_{\sigma}\right) \geq \sum \lambda_{\sigma} N(x_{\sigma}) = \left(\sum \lambda_{\sigma}\right) N(x) = N(x).$$

This finishes the proof. □

2.2 Proof of Theorem 3

Fix $p \geq 1$ and let $q \in [1, \infty]$ be its conjugate, $\frac{1}{p} + \frac{1}{q} = 1$. Let B_q be the closed unit ball in L_q (of the underlying probability space with the norm $\|Y\|_q = (\mathbb{E}|Y|^q)^{1/q}$). For $t \in \mathbb{R}^n$, we denote

$$X_t = \sum_j e^{t_j} \varepsilon_j.$$

Thanks to Hölder's inequality, we have

$$\|X_t\|_p = \max_{Y \in B_q} \mathbb{E} X_t Y, \qquad t \in \mathbb{R}^n,$$

with the maximum attained at

$$Y_*(t) = \frac{1}{\|X_t\|_p^{p-1}} \operatorname{sgn}(X_t) |X_t|^{p-1}.$$

The main idea is to consider the subset A_q of B_q of random variables with nonnegative correlations with all ε_j,

$$A_q = \left\{Y \in B_q,\ \mathbb{E}[Y \varepsilon_j] \geq 0,\ j = 1, \ldots, n\right\}.$$

Claim. $Y_*(t) \in A_q$, for every $t \in \mathbb{R}^n$.

As a result,

$$\|X_t\|_p = \max_{Y \in A_q} \mathbb{E} X_t Y, \qquad t \in \mathbb{R}^n,$$

which allows to finish the proof in one line. We have,

$$\frac{1}{p}\Phi(t) = \log \mathbb{E}\|X_t\|_p = \max_{Y \in A_q} \log \mathbb{E} X_t Y = \max_{Y \in A_q} \log \left(\sum_{j=1}^n e^{t_j} \mathbb{E}[Y \varepsilon_j] \right)$$

Functions $t \mapsto \log \sum_{j=1}^n e^{t_j} \mathbb{E}[Y\varepsilon_j]$ are convex (as sums of log-convex functions are log-convex), so their pointwise maximum over $Y \in A_q$ is also convex.

Proof of the claim. Let $f(x) = \operatorname{sgn}(x)|x|^{p-1}$ which is nondecreasing. Fix $j \leq n$ and note that evaluating the expectation against ε_j gives

$$\|X_t\|_p^{p-1} \mathbb{E}[Y_*(t)\varepsilon_j] = \mathbb{E}[f(X_t)\varepsilon_j] = \frac{1}{2}\mathbb{E}\left[f\left(e^{t_j} + \sum_{i \neq j} e^{t_i}\varepsilon_i \right) - f\left(-e^{t_j} + \sum_{i \neq j} e^{t_i}\varepsilon_i \right) \right].$$

The square bracket is nonnegative as $f(v+u) \geq f(v-u)$ for every $u \geq 0$ and $v \in \mathbb{R}$, by monotonicity. □

Remark 4. We have crucially used that the class of log-convex functions is stable under summation, or more generally, if $\{f_\alpha(x)\}_{\alpha \in \mathcal{A}}$ is a family of log-convex functions on, say $\mathbb{R}^n$, then the function

$$x \mapsto \int_{\mathcal{A}} f_\alpha(x) \mathrm{d}\mu(\alpha) \tag{5}$$

is also log-convex on $\mathbb{R}^n$, where μ is a nonnegative measure on $\mathcal{A}$. This readily follows from Hölder's inequality. As a result, Theorem 3 instantly extends to sums of independent symmetric random variables (a random variable X is symmetric if $-X$ and X have the same distribution).

Corollary 5. *Let $X_1, X_2, \ldots$ be independent symmetric random variables. For every $n \geq 1$ and $p \geq 1$, the function*

$$\Phi(t_1, \ldots, t_n) = \log \mathbb{E}\left| e^{t_1} X_1 + \cdots + e^{t_n} X_n \right|^p$$

is convex on $\mathbb{R}^n$.

For the proof, note that by the symmetry of the X_j, they have the same distribution as $\varepsilon_j|X_j|$, respectively, where $\varepsilon_1,\ldots,\varepsilon_n$ are independent Rademacher random variables (independent of the X_j). Thus, it suffices to use (5) with μ given by the distribution of $(|X_1|,\ldots,|X_n|)$.

3 Concluding remarks

3.1 Monotonicity under ℓ_∞ normalisation

Aliev in Lemma 2 in [1] showed that for the unit cube Q_n, its Busemann norm $N_{Q_n}(x)=\frac{|x|}{\mathrm{vol}_{n-1}(Q_n\cap x^\perp)}$ (see Theorem 2) is maximised over the unit ℓ_∞-sphere at its vertices. Since the maximum of a convex function over a convex body is attained at an extreme point, Aliev's lemma extends in such a statement to all origin-symmetric convex bodies. Moreover, since an even convex function on the real line is nondecreasing, for 1-symmetric bodies we obtain a stronger monotonicity property.

Theorem 6. *Let K be a 1-symmetric convex body in $\mathbb{R}^n$. Then the function $x\mapsto N_K(x)$ defined in (3) is monotone with respect to each coordinate on $\mathbb{R}^n_+$. In particular,* $\max_{x\in[0,1]^n} N_K(x)=N_K(1,\ldots,1)$.

3.2 Dual logarithmic Brunn-Minkowski inequality

Saroglou's dual log Brunn-Minkowski inequality, Theorem 6.2 from [14], essentially states that the $(n-1)$-volume of the polytope

$$P_t=\mathrm{conv}\left\{\pm e^{t_j}\mathrm{Proj}_{(1,\ldots,1)^\perp}e_j,\ j\le n\right\}=\mathrm{Proj}_{(1,\ldots,1)^\perp}\mathrm{conv}\{\pm e^{t_j}e_j,\ j\le n\}$$

is log-convex. As usual, for a subspace H in $\mathbb{R}^n$, Proj_H denotes the orthogonal projection onto H. On the other hand, the 2^n facets of the stretched cross-polytope $\mathrm{conv}\{\pm e^{t_j}e_j\}$ are all congruent with the outer normal vectors $\left(\sum_j e^{-2t_j}\right)^{-1/2}[\varepsilon_j e^{-t_j}]_{j=1}^n$, $\varepsilon\in\{-1,1\}$ and the $(n-1)$-volume $n\left(\sum_j e^{-2t_j}\right)^{1/2}e^{\sum t_j}$, so from Cauchy's formula (see for instance, [9, 13]), we get

$$\mathrm{vol}_{n-1}(P_t) = 2^{n-1} n e^{\sum t_j} \mathbb{E} \left| \sum_j e^{-t_j} \varepsilon_j \right|.$$

Thus the convexity of the function

$$t \mapsto \log \mathbb{E} \left| \sum_j e^{t_j} \varepsilon_j \right| \tag{6}$$

is a special case of Saroglou's result. In that sense, Theorem 3 can be viewed as a probabilistic extension of the dual logarithmic Brunn-Minkowski inequality. In analogy to Theorem 3, we thus conjecture that the following extension of (4) holds: for every $0 < q \leq 1$, the function

$$t \mapsto -\log \left(\mathbb{E} \left[\left| e^{t_1} \xi_1 + \cdots + e^{t_n} \xi_n \right|^{-q} \right] \right)$$

is convex on $\mathbb{R}^n$.

3.3 A vector-valued extension

As an immediate corollary to Theorem 3, we obtain its extension to vector-valued coefficients in Hilbert space.

Corollary 7. *Let $\varepsilon_1, \varepsilon_2, \ldots$ be independent Rademacher random variables. Let H be a separable Hilbert space with norm $\|\cdot\|$. Let $p \geq 1$ and $v_1, \ldots, v_n$ be vectors in H. Then*

$$\Phi(t_1, \ldots, t_n) = \log \mathbb{E} \left\| e^{t_1} \varepsilon_1 v_1 + \cdots + e^{t_n} \varepsilon_n v_n \right\|^p$$

is convex on $\mathbb{R}^n$.

Proof. We use a standard embedding (see, e.g. Remark 3 in [15]): we fix an orthonormal basis in H, say $(u_k)_{k \geq 1}$, take i.i.d. standard Gaussian random variables $g_1, g_2, \ldots$, independent of the ε_j and set $G = \sum_{k\geq 1} g_k u_k$ to have

$$\|x\|^p = \frac{1}{\mathbb{E}|g_1|^p}\mathbb{E}\,|\langle x, G\rangle|^p\,, \qquad x \in H.$$

This gives that

$$\Phi(t_1, \ldots, t_n) = -\log \mathbb{E}|g_1|^p + \log \mathbb{E}_G\mathbb{E}_\varepsilon \left|\sum_{j=1}^n e^{t_j}\,\langle v_j, G\rangle\,\varepsilon_j\right|^p.$$

The result follows from Theorem 3, for conditioned on the value of G, the function $t \mapsto \mathbb{E}_\varepsilon \left|\sum_{j=1}^n e^{t_j}\,\langle v_j, G\rangle\,\varepsilon_j\right|^p$ is log-convex and sums of log-convex functions are log-convex. □

3.4 A Representation as a maximum

We finish with an elementary representation of the L_1 norm of Rademacher sums as a maximum of linear forms with nonnegative ordered coefficients. This besides being of independent interest gives an alternative proof of the convexity of (6), as explained at the end of this subsection. For $n \geq 1$, we let

$$T_n = \{x \in \mathbb{R}^n,\ x_1 \geq x_2 \geq \ldots x_n \geq 0\}$$

be the cone in $\mathbb{R}^n$ of nonincreasing nonnegative sequences.

Lemma 8. *For every $n \geq 1$, there is a finite subset A_n of T_n such that for all $x \in T_n$, we have*

$$\mathbb{E}\left|\sum_{j=1}^n x_j\varepsilon_j\right| = \max_{a\in A_n}\ \sum_{j=1}^n a_j x_j.$$

Proof. Changing the order of summation, we write

$$\mathbb{E}\left|\sum_{j=1}^n x_j\varepsilon_j\right| = \mathbb{E}\left[\operatorname{sgn}\left(\sum_{j=1}^n x_j\varepsilon_j\right)\left(\sum_{j=1}^n x_j\varepsilon_j\right)\right] = \sum_{j=1}^n x_j\mathbb{E}\left[\varepsilon_j\operatorname{sgn}\left(\sum_{i=1}^n x_i\varepsilon_i\right)\right],$$

where we use the standard signum function, $\operatorname{sgn}(t) = |t|/t$, $t \neq 0$, $\operatorname{sgn}(0) = 0$ which is odd and nondecreasing. It is thus natural to define the function $\alpha =$

$(\alpha_1, \dots, \alpha_n) \colon \mathbb{R}^n \to \mathbb{R}^n$,

$$\alpha_j(x) = \mathbb{E}\left[\varepsilon_j \operatorname{sgn}\left(\sum_{i=1}^n x_i \varepsilon_i\right)\right].$$

Note that since $\operatorname{sgn}(\cdot)$ is odd, we have

$$\alpha_j(x) = \mathbb{E}\operatorname{sgn}\left(x_j + \sum_{i \neq j} x_i \varepsilon_i\right), \qquad x \in \mathbb{R}^n.$$

We set

$$A_n = \alpha(T_n)$$

and to finish the proof, we claim that

(1) A_n is a finite set,

(2) $A_n \subset T_n$,

(3) $\mathbb{E}|\sum_{j=1}^n x_j \varepsilon_j| = \max_{a \in A_n} \ \sum_{j=1}^n a_j x_j$, for every $x \in T_n$.

Claim (1) holds because $\alpha_j(x)$ takes only finitely many values (for any x, $\alpha_j(x)$ is a sum of 2^n terms, each equal to $\pm\frac{1}{2^n}$ or 0).

To show (2), we fix $x \in T_n$ and $1 \leq k \leq n-1$. To argue that $\alpha_k(x) \geq \alpha_{k+1}(x)$, we write

$$\begin{aligned} \alpha_k(x) - \alpha_{k+1}(x) &= \mathbb{E}\operatorname{sgn}\left(x_k + \varepsilon_{k+1}x_{k+1} + \sum_{i \neq k,k+1} x_i\varepsilon_i\right) - \mathbb{E}\operatorname{sgn}\left(x_{k+1} + \varepsilon_k x_k + \sum_{i \neq k,k+1} x_i\varepsilon_i\right) \\ &= \mathbb{E}\left[\operatorname{sgn}\left(x_k - x_{k+1} + \sum_{i \neq k,k+1} x_i\varepsilon_i\right) - \mathbb{E}\operatorname{sgn}\left(x_{k+1} - x_k + \sum_{i \neq k,k+1} x_i\varepsilon_i\right)\right] \end{aligned}$$

and the monotonicity of $\operatorname{sgn}(\cdot)$ finishes the argument. We also need to show that $\alpha_n(x) \geq 0$. Taking the expectation with respect to ε_n in the definition of α_n, we have

$$\alpha_n(x) = \frac{1}{2}\mathbb{E}\left[\operatorname{sgn}\left(x_n + \sum_{i<n} x_i\varepsilon_i\right) - \operatorname{sgn}\left(-x_n + \sum_{i<n} x_i\varepsilon_i\right)\right]$$

and the expression inside the expectation is nonnegative because $\operatorname{sgn}(v+u) \geq \operatorname{sgn}(v-u)$ for every $u \geq 0$ and $v \in \mathbb{R}$, by monotonicity.

Finally, to prove (3), we fix $x \in T_n$, take arbitrary $a \in A_n$, say $a = \alpha(y)$ with $y \in T_n$ and note that

$$\begin{aligned}
\sum_j a_j x_j = \sum_j \alpha_j(y) x_j &= \sum_j \mathbb{E}\left[\varepsilon_j \operatorname{sgn}\left(\sum_i y_i \varepsilon_i\right)\right] x_j \\
&= \mathbb{E}\left[\operatorname{sgn}\left(\sum_i y_i \varepsilon_i\right)\left(\sum_j x_j \varepsilon_j\right)\right] \\
&\leq \mathbb{E}\left|\sum_j x_j \varepsilon_j\right|
\end{aligned}$$

proving that $\max_{a \in A_n} a_j x_j \leq \mathbb{E}|\sum x_j \varepsilon_j|$ with the equality plainly attained for $a = \alpha(x)$. □

If we now account for all possible orderings by taking the maximum over all permutations in the symmetric group S_n on $\{1, \ldots, n\}$, we obtain a representation for arbitrary coefficients.

Corollary 9. *Let $n \geq 1$ and let A_n be the finite subset provided by Lemma 8. For every $x \in \mathbb{R}_+^n$, we have*

$$\mathbb{E}\left|\sum_{j=1}^n x_j \varepsilon_j\right| = \max_{a \in A_n, \sigma \in S_n} \sum_{j=1}^n a_j x_{\sigma(j)}.$$

Proof. Fix $x \in \mathbb{R}_+^n$ and let σ^* be a permutation such that

$$x_{\sigma^*(1)} \geq \cdots \geq x_{\sigma^*(n)}.$$

By Lemma 8,

$$\mathbb{E}\left|\sum_{j=1}^n x_j \varepsilon_j\right| = \mathbb{E}\left|\sum_{j=1}^n x_{\sigma^*(j)} \varepsilon_j\right| = \max_{a \in A_n} \sum_{j=1}^n a_j x_{\sigma^*(j)}$$

Moreover, by the rearrangement inequality, for an arbitrary permutation σ and arbitrary $a \in A_n$ we get

$$\sum_{j=1}^{n} a_j x_{\sigma^*(j)} \geq \sum_{j=1}^{n} a_j x_{\sigma(j)}$$

since both sequences (a_j) and $(x_{\sigma^*(j)})$ are nonincreasing. This finishes the proof. □

To see that the function in (6) is convex, note that from Corollary 9, we have

$$\Phi(t_1, \ldots, t_n) = \max_{\sigma \in S_n, a \in A_n} \log \sum_{j=1}^{n} a_j e^{t_{\sigma(j)}}.$$

For a fixed $a \in A_n$ and $\sigma \in S_n$, the function

$$\log \sum_{j=1}^{n} a_j e^{t_{\sigma(j)}}$$

is convex (as sums of log-convex functions are log-convex). Thus so is their pointwise maximum. □

References

[1] Aliev, I., Siegel's lemma and sum-distinct sets. Discrete Comput. Geom. 39 (2008), no. 1-3, 59–66.

[2] Aliev, I., On the volume of hyperplane sections of a d-cube. Acta Math. Hungar. 163 (2021), no. 2, 547–551.

[3] Ball, K., Logarithmically concave functions and sections of convex sets in R^n. Studia Math. 88 (1988), no. 1, 69–84.

[4] Bárány, I., Frankl, P., How (not) to cut your cheese. Amer. Math. Monthly 128 (2021), no. 6, 543–552.

[5] Bárány, I., Frankl, P., Cells in the box and a hyperplane. J. Eur. Math. Soc. (JEMS) 25 (2023), no. 7, 2863–2877.

[6] Bhatia, R., Matrix analysis. Graduate Texts in Mathematics, 169. Springer-Verlag, New York, 1997.

[7] Böröczky, K., Lutwak, E., Yang, D., Zhang, G., The log-Brunn-Minkowski inequality. Adv. Math. 231 (2012), no. 3-4, 1974–1997.

[8] Busemann, H., A theorem on convex bodies of the Brunn-Minkowski type. Proc. Nat. Acad. Sci. U.S.A. 35 (1949), 27–31.

[9] Gardner, R. J., Geometric tomography. Second edition. Encyclopedia of Mathematics and its Applications, 58. Cambridge University Press, New York, 2006.

[10] König, H., Koldobsky, A., Volumes of low-dimensional slabs and sections in the cube, Adv. Appl. Math. 47(4) (2011), 894–907.

[11] Milman, V. D., Pajor, A., Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space. Geometric aspects of functional analysis (1987–88), 64–104, Lecture Notes in Math., 1376, Springer, Berlin, 1989.

[12] Nayar, P., Tkocz, T., On a convexity property of sections of the cross-polytope. Proc. Amer. Math. Soc. 148 (2020), no. 3, 1271–1278.

[13] Nayar, P., Tkocz, T., Extremal sections and projections of certain convex bodies: a survey. Harmonic analysis and convexity, 343–390, Adv. Anal. Geom., 9, De Gruyter, Berlin, 2023.

[14] Saroglou, C., More on logarithmic sums of convex bodies. Mathematika 62 (2016), no. 3, 818–841.

[15] Szarek, S. J., On the best constants in the Khinchin inequality. Studia Math. 58 (1976), no. 2, 197–208.