Concentration inequalities and geometry of convex bodies

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Abstract

Our goal is to write an extended version of the notes of a course given by Olivier Guédon at the Polish Academy of Sciences from April 11-15, 2011. The course is devoted to the study of concentration inequalities in the geometry of convex bodies, going from the proof of Dvoretzky's theorem due to Milman [75] until the presentation of a theorem due to Paouris [78] telling that most of the mass of an isotropic convex body is "contained" in a multiple of the Euclidean ball of radius the square root of the ambient dimension. The purpose is to cover most of the mathematical stuff needed to understand the proofs of these results. On the way, we meet different topics of functional analysis, convex geometry and probability in Banach spaces. We start with harmonic analysis, the Brascamp-Lieb inequalities and its geometric consequences. We go through some functional inequalities like the functional Prékopa-Leindler inequality and the well-known Brunn-Minkowski inequality. Other type of functional inequalities have nice geometric consequences, like the Busemann Theorem, and we will present some of them. We continue with the Gaussian concentration inequalities and the classical proof of Dvoretzky's theorem. The study of the reverse Hölder inequalities (also called reverse Lyapunov's inequalities) is very developed in the context of log-concave or γ-concave functions. Finally, we present a complete proof of the result of Paouris [78]. We will need most of the tools introduced during the previous lectures. The Dvoretzky theorem, the notion of Z_p bodies and the reverse Hölder inequalities are the fundamentals of this proof. There are classical books or surveys about these subjects and we refer to [8, 9, 13, 48, 49, 55, 30, 80, 27] for further readings. The notes are accessible to people with classical knowledge about integration, functional and/or harmonic analysis and probability.

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1 Introduction

In harmonic analysis, Young's inequalities tell that for a locally compact group G equipped with its Haar measure, if $1/p + 1/q = 1 + 1/s$ then

$$\forall f \in L_p(G), g \in L_q(G), \quad \|f \star g\|_s \leq \|f\|_p \|g\|_q.$$

The constant 1 is optimal for compact groups such that constant functions belong to each $L_p(G)$. However, it is not optimal for example in the real case. During the seventies, Beckner [14] and Brascamp-Lieb [26] proved that the extremal functions in Young's inequality are among Gaussian densities. We discuss the geometric version of these inequalities introduced by Ball [7]. The problem of computing the value of the integrals for the maximizers disappears when we write these inequalities in a geometric context. The proof can be done via the transport argument that we will present. The geometric applications of this result are that the cube, among symmetric convex bodies, has several extremal properties. Indeed, Ball [7] proved a reverse isoperimetric inequality, namely that for every centrally symmetric convex body K in $\mathbb{R}^n$, there exists a linear transformation $\widetilde{K}$ of K such that

$$\text{Vol } \widetilde{K} = \text{Vol } B_\infty^n \quad \text{and} \quad \text{Vol } \partial \widetilde{K} \leq \text{Vol } \partial B_\infty^n.$$

Moreover, in the case of random Gaussian averages, Schechtman and Schmuckenschläger [86] proved that for every centrally symmetric convex body K in $\mathbb{R}^n$ which is in the so-called John position, $\mathbb{E}\|(g_1, \dots, g_n)\|_K \geq \mathbb{E}|(g_1, \dots, g_n)|_\infty$ where $g_1, \dots, g_n$ are independent Gaussian standard random variables.

Another powerful inequality in convex geometry is the Prékopa-Leindler inequality [82]. This is a functional version of the Brunn-Minkowski inequality which tells that for any non-empty compact sets $A, B \subset \mathbb{R}^n$

$$\text{vol}(A + B)^{1/n} \geq \text{vol}(A)^{1/n} + \text{vol}(B)^{1/n}.$$

We prove the Prékopa-Leindler inequality and we discuss a modified version of this inequality introduced by Ball [6], see also [24]. Ball [6] used it to create a bridge between probability and convex geometry, namely that one can associate a convex body with any log-concave measure.

1.1 Theorem. *Suppose $f : \mathbb{R}^n \to \mathbb{R}_+ \in L_1(\mathbb{R}^n)$ is an even log-concave function and $p > -1$. Then*

$$\|x\| = \begin{cases} \left(\int_0^\infty r^p f(rx) dr \right)^{-1/p+1} & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$$

defines a norm on $\mathbb{R}^n$.

The result is seen as a generalisation of Busemann theorem [29]. Some properties of these bodies will be studied in Section 6.

Dvoretzky's Theorem tells that ℓ_2 is finitely representable in any infinite dimensional Banach space. Its quantified version due to Milman [80] is one of the fundamental result of the local theory of Banach spaces.

1.2 Theorem. *Let K be a symmetric convex body such that $K \subset B_2^n$. Define*

$$M^\star(K) = \int_{S^{n-1}} h_K(\theta) d\sigma(\theta).$$

Then for all $\varepsilon > 0$ there exists a vector subspace E of dimension

$$k = k^*(K) = \lfloor cn(M^\star(K))^2 \varepsilon^2 / \log(1/\varepsilon) \rfloor$$

such that

$$(1-\varepsilon) M^\star(K) P_E B_2^n \subset P_E K \subset (1+\varepsilon) M^\star(K) P_E B_2^n.$$

Instead of using the concentration of measure on the unit Euclidean sphere, this can be proved via the use of Gaussian operators. We will present some classical concentration inequalities of a norm of a Gaussian vector following the ideas of Maurey and Pisier [80]. The argument of the proof of Dvoretzky's theorem is now standard and is done in three steps: a concentration inequality for an individual vector of the unit sphere, a net argument and discretisation of the sphere, a union bound and optimisation of the parameters.

The subject of the inverse Hölder inequalities is very wide. In the context of log-concave or s-concave measures, major tools were developed by Borell [21, 20]. In particular, he proved that for every log-concave function $f : [0, \infty) \to \mathbb{R}_+$, the function

$$p \mapsto \frac{1}{\Gamma(p+1)} \int_0^\infty t^p f(t) dt$$

is log-concave on $(-1, +\infty)$. For $p \geq 1$, the Z_p-body associated with a log-concave density f is defined by it support function

$$h_{Z_p(f)}(\theta) = \left(\int \langle x, \theta \rangle_+^p f(x) dx \right)^{1/p},$$

where $\langle x, \theta \rangle_+$ is the positive part of $\langle x, \theta \rangle$. We present some basic properties of these bodies. It will be of particular interest to understand the behaviour of the bodies $Z_p(\pi_E(f))$ where $\pi_E(f)$ is the marginal density of f on a k-dimensional subspace E. The inverse Hölder inequalities give some information and we will try to explain how it reflects geometric properties of the density f.

The goal of the last Section is to present a probabilistic version of Paouris theorem [78] that appeared in [2].

1.3 Theorem. *There exists a constant C such that for any random vector X distributed according to a log-concave probability measure on $\mathbb{R}^n$, we have*

$$(\mathbb{E}|X|_2^p)^{1/p} \leq C\,(\mathbb{E}|X|_2 + \sigma_p(X))$$

for all $p \geq 1$, where $\sigma_p(X) = \sup_{\theta \in S^{n-1}} \mathbb{E}\,\langle X, \theta\rangle_+^p$ is the weak p-th moment associated with X.

Moreover, if X is such that for any $\theta \in S^{n-1}$, $\mathbb{E}\,\langle X, \theta\rangle^2 = 1$, then for any $t \geq 1$,

$$\mathbb{P}(|X|_2 \geq c\,t\sqrt{n}) \leq \exp(-t\sqrt{n}),$$

where c is a universal constant.

Most of the tools presented in the first lectures are needed to make this proof : Dvoretzky's theorem, Z_p-bodies, the inverse Hölder inequalities. The sketch of the proof is the following. Let $G \sim \mathcal{N}(0, \mathrm{Id})$ be a standard Gaussian random vector in $\mathbb{R}^n$. Observe that for any random vector X distributed with a log-concave density f,

$$\begin{aligned}(\mathbb{E}|X|_2^p)^{1/p} &= (\gamma_p^+)^{-1}\,(\mathbb{E}_X\mathbb{E}_G\langle X, G\rangle_+^p)^{1/p} \\ &= (\gamma_p^+)^{-1}\left(\mathbb{E}_G h_{Z_p(f)}(G)^p\right)^{1/p},\end{aligned}$$

where for a standard Gaussian random variable $g \sim \mathcal{N}(0,1)$, $\gamma_p^+ = (\mathbb{E}g_+^p)^{1/p}$. By a Gaussian concentration inequality, we see that for any $1 \leq p \leq ck^*(Z_p(f))$,

$$\left(\mathbb{E}h_{Z_p(f)}(G)^p\right)^{1/p} \approx \mathbb{E}h_{Z_p(f)}(G) = M^*(Z_p(f))\mathbb{E}|G|_2,$$

where $k^*(Z_p(f))$ is the Dvoretzky dimension of the convex $Z_p(f)$. Looking at the conclusion of Dvoretzky's theorem, we also observe that $M^*(Z_p(f))$ is the $\frac{1}{k}$-th power of the volume of most of the k-dimensional projection of $Z_p(f)$ where $k \leq k^*(Z_p(f))$. It remains to study the volume of these projections. For any k dimensional subspace E, let $\pi_E f$ denote the marginal of the density f on E, that is

$$\forall x \in E,\ \pi_E f(x) = \int_{E^\perp} f(x+y)dy.$$

By the Prékopa-Leindler inequality, $\pi_E f$ is still log-concave on E. We can prove that for any $p \geq 1$ and any k-dimensional subspace E

$$P_E(Z_p(f)) = Z_p(\pi_E f) = Z_p(K_{k+p}(\pi_E f)),$$

where $K_{k+p}(\pi_E f)$ is the convex body whose norm is

$$\|x\|_{K_{k+p}(\pi_E f)} = \left((k+p)\int_0^\infty t^{k+p-1}\pi_E f(tx)dt\right)^{-\frac{1}{k+p}}.$$

In a log-concave setting, we will see that for $p \geq k$, $Z_p(K_{k+p}(\pi_E f))$ is "approximately" $K_{k+p}(\pi_E f)$ so that the $\frac{1}{k}$-th power of the volume of $P_E(Z_p(f))$ is approximately the $\frac{1}{k}$-th power of the volume of $K_{k+p}(\pi_E f)$. The reverse Hölder inequalities will give several properties that will lead to the conclusion.

Besides the standard notation, we adopt throughout the notes the common convention that universal constants sometimes change from line to line.

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2 Brascamp-Lieb inequalities in a geometric context

2.1 Motivation and formulation of the inequality

Let G be a locally compact group with Haar measure μ. Let $p, q, s \geq 1$ be such that $1/p + 1/q = 1 + 1/s$, let $f \in L_p(G,\mu)$ and $g \in L_q(G,\mu)$. Then we have the following Young's inequality

$$\|f \star g\|_s \leq \|f\|_p \|g\|_q , \tag{2.1}$$

where

$$(f \star g)(x) = \int_G f(xy^{-1})g(y) \,\mathrm{d}\mu(y).$$

The constant 1 in (2.1) is optimal when constant functions belong to $L_p(G)$, $p \geq 1$, but it is not optimal when $G = \mathbb{R}$ and μ is the Lebesgue measure. In the seventies, Beckner and independently Brascamp and Lieb proved that in $\mathbb{R}$ the *"equality case"* is achieved for sequences of functions f_n and g_n with Gaussian densities, i.e. functions of the form

$$h_a(x) = \sqrt{a/\pi} e^{-ax^2}.$$

Note that if $1/r + 1/s = 1$ then

$$\|f \star g\|_s = \sup_{\substack{h \in L_r(\mathbb{R}) \\ \|h\|_r \leq 1}} \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-y)g(y)h(x) \,\mathrm{d}y \,\mathrm{d}x$$

and we have $f \in L_p(\mathbb{R})$, $g \in L_q(\mathbb{R})$, $h \in L_r(\mathbb{R})$ with $\frac{1}{r} + \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1 + \frac{1}{s} = 2$. Let $v_1 = (1,-1)$, $v_2 = (0,1)$ and $v_3 = (1,0)$. Then

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(x-y)g(y)h(x) \,\mathrm{d}y \,\mathrm{d}x = \int_{\mathbb{R}^2} f(\langle X, v_1\rangle)g(\langle X, v_2\rangle)h(\langle X, v_3\rangle) \,\mathrm{d}X.$$

This is a type of expression studied by Brascamp and Lieb. Namely, they prove

2.1 Theorem. *Let $n, m \geq 1$ and let $p_1, \ldots, p_m > 0$ be such that $\sum_{i=1}^m \frac{1}{p_i} = n$. If $v_1, \ldots, v_m \in \mathbb{R}^n$ and $f_1, \ldots, f_m : \mathbb{R} \to \mathbb{R}_+$ then*

$$\frac{\int_{\mathbb{R}^m} \prod_{i=1}^m f_i(\langle v_i, x\rangle) \,\mathrm{d}x}{\prod_{i=1}^m \|f_i\|_{p_i}}$$

is "maximized" *when $f_1, \ldots, f_m$ are Gaussian densities. However, the supremum may not be attained in the sense that one has to consider Gaussian densities f_a with $a \to 0$.*

In this context, it remains to compute the constants for the extremal Gaussian densities which is not so easy. In a geometric setting we have a version of the Brascamp-Lieb inequality due to Ball [7].

2.2 Theorem. *Let $n, m \geq 1$ and let $u_1, \ldots, u_m \in S^{n-1}$, $c_1, \ldots, c_m > 0$ be such that* $\mathrm{Id} = \sum_{j=1}^m c_j u_j \otimes u_j$. *If $f_1, \ldots, f_m : \mathbb{R} \to \mathbb{R}_+$ are integrable functions then*

$$\int_{\mathbb{R}^n} \prod_{j=1}^m (f_j(\langle x, u_j \rangle))^{c_j} \, \mathrm{d}x \leq \prod_{j=1}^m \left(\int_{\mathbb{R}} f_j \right)^{c_j}. \tag{2.2}$$

2.3 Remark. The condition

$$\mathrm{Id} = \sum_{j=1}^m c_j u_j \otimes u_j \tag{2.3}$$

means that

$$\forall x \in \mathbb{R}^n, \quad x = \sum_{j=1}^m c_j \langle x, u_j \rangle u_j$$

and is equivalent to

$$\forall x \in \mathbb{R}^n, \quad |x|_2^2 = \sum_{j=1}^m c_j \langle x, u_j \rangle^2 .$$

We can easily construct examples of vectors satisfying condition (2.3). Let H be an n-dimensional subspace of $\mathbb{R}^m$. Let $e_1, \ldots, e_m$ be the standard orthonormal basis in $\mathbb{R}^m$ and let $P : \mathbb{R}^m \to H$ be the orthogonal projection onto H. Clearly, $\mathrm{Id}_{\mathbb{R}^m} = \sum_{j=1}^m e_j \otimes e_j$ and $x = \sum_{j=1}^m \langle x, e_j \rangle e_j$, hence $Px = \sum_{j=1}^m \langle x, e_j \rangle Pe_j$. If $x \in H$ then $Px = x$ and $\langle x, e_j \rangle = \langle Px, e_j \rangle = \langle x, Pe_j \rangle$, therefore $x = \sum_{j=1}^m \langle x, Pe_j \rangle Pe_j$. Thus $\mathrm{Id}_{H \approx \mathbb{R}^n} = \sum_{j=1}^m c_j u_j \otimes u_j$, where $c_j = |Pe_j|^2$ and $u_j = Pe_j / |Pe_j|$.

2.4 Remark. Let $f_j(t) = e^{-\alpha t^2}$ for $1 \leq j \leq m$. If (2.3) is satisfied then

$$\prod_{j=1}^m (f_j(\langle x, u_j \rangle))^{c_j} = \exp\left(-\sum_{j=1}^m \alpha c_j \langle x, u_j \rangle^2 \right) = \exp(-\alpha |x|_2^2).$$

Thus,

$$\begin{aligned} \int_{\mathbb{R}^n} \prod_{j=1}^m (f_j(\langle x, u_j \rangle))^{c_j} \, \mathrm{d}x &= \int_{\mathbb{R}^n} \exp(-\alpha |x|_2^2) \, \mathrm{d}x = \left(\int_{\mathbb{R}} \exp(-\alpha t^2) \, \mathrm{d}t \right)^n \\ &= \prod_{j=1}^m \left(\int_{\mathbb{R}} \exp(-\alpha t^2) \, \mathrm{d}t \right)^{c_j} = \prod_{j=1}^m \left(\int_{\mathbb{R}} f_j \right)^{c_j}, \end{aligned}$$

since we have

$$n = \mathrm{tr}(\mathrm{Id}) = \sum_{j=1}^m c_j \mathrm{tr}(u_j \otimes u_j) = \sum_{j=1}^m c_j |u_j|_2^2 = \sum_{j=1}^m c_j.$$

Therefore we have equality in (2.2) when f_j's are identical Gaussian densities.

2.2 The proof

We start the proof of Theorem 2.2 with a simple lemma.

2.5 Lemma. *Suppose $u_1, \ldots, u_m \in S^{n-1}$ and $c_1, \ldots, c_m$ are positive numbers. Assume that $\mathrm{Id} = \sum_{j=1}^m c_j u_j \otimes u_j$. Then*

(1) If $x = \sum_{j=1}^m c_j \theta_j u_j$ for some numbers $\theta_1, \ldots, \theta_m$, then $|x|_2^2 \le \sum_{j=1}^m c_j \theta_j^2$.

(2) For all $T \in L(\mathbb{R}^n)$ we have

$$|\det T| \le \prod_{j=1}^m |Tu_j|_2^{c_j}$$

(a generalisation of Hadamard's inequality).

(3) For all $\alpha_1, \ldots, \alpha_m > 0$ we have

$$\det\left(\sum_{j=1}^m c_j \alpha_j u_j \otimes u_j\right) \ge \prod_{j=1}^m \alpha_j^{c_j}.$$

Moreover, if $\alpha_1 = \ldots = \alpha_m$, then equality holds.

Proof. (1) Using the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} |x|_2^2 = \langle x, x\rangle &= \left\langle \sum_{j=1}^m c_j \theta_j u_j, x \right\rangle = \sum_{j=1}^m c_j \theta_j \langle u_j, x\rangle \\ &\le \left(\sum_{j=1}^m c_j \theta_j^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^m c_j \langle u_j, x\rangle^2\right)^{\frac{1}{2}} = \left(\sum_{j=1}^m c_j \theta_j^2\right)^{\frac{1}{2}} |x|_2. \end{aligned}$$

(2) We can assume that T is symmetric and positive definite. Indeed, since $T^\star T$ is symmetric, for any $T \in GL_n(\mathbb{R})$ we have the decomposition $T^\star T = U^\star D U$, where U is orthogonal and D is diagonal. Let $S = U^\star D^{\frac{1}{2}} U$. Clearly, $S^2 = T^\star T$ and S is symmetric and positive definite. Suppose we can show (2) for S. Then we have

$$|\det S| = \sqrt{\det D} = \sqrt{\det T^\star T} = |\det T|$$

and

$$|Tu_j|_2^2 = \langle Tu_j, Tu_j\rangle = \langle u_j, T^\star T u_j\rangle = \langle u_j, S^2 u_j\rangle = \langle Su_j, Su_j\rangle = |Su_j|_2^2.$$

Thus (2) is also true for T.

Assume that T is symmetric and positive definite. Then there exist $\lambda_1, \dots, \lambda_n > 0$ and an orthonormal basis $v_1, \dots, v_n$ of $\mathbb{R}^n$ such that

$$T = \sum_{i=1}^{n} \lambda_i v_i \otimes v_i.$$

Clearly, $Tu_j = \sum_{i=1}^{n} \lambda_i \langle u_j, v_i \rangle v_i$ and therefore

$$|Tu_j|_2^2 = \sum_{i=1}^{n} \lambda_i^2 \langle u_j, v_i \rangle^2 .$$

Since $|u_j|_2 = 1$, we have $\sum_{i=1}^{n} \langle u_j, v_i \rangle^2 = 1$. Let $\lambda_i^2 = a_i \geq 0$ and $p_i = \langle u_j, v_i \rangle^2$. Then $\sum_{i=1}^{n} p_i = 1$ and therefore by the AM–GM inequality, we get

$$\sum_{i=1}^{n} a_i p_i \geq \prod_{i=1}^{n} a_i^{p_i},$$

which means that

$$|Tu_j|_2^2 \geq \prod_{i=1}^{n} \lambda_i^{2|\langle u_j, v_i \rangle|^2}.$$

We obtain

$$\prod_{j=1}^{m} |Tu_j|_2^{c_j} \geq \prod_{i=1}^{n} \lambda_i^{\sum_{j=1}^{m} c_j |\langle u_j, v_i \rangle|^2} = \prod_{i=1}^{n} \lambda_i = \det T,$$

as $\sum_{j=1}^{m} c_j | \langle u_j, v_i \rangle |^2 = |v_i|_2^2 = 1$.

(3) We prove that for all symmetric positive definite matrices we have

$$(\det S)^{1/n} = \min_{T\colon \det T = 1} \frac{(\operatorname{tr} TST^\star)}{n}. \tag{2.4}$$

If $\lambda_1, \dots, \lambda_n \geq 0$ are the eigenvalues of the symmetric and positive definite matrix $TST^\star$ then

$$\frac{(\operatorname{tr} TST^\star)}{n} = \frac{1}{n} \sum_{i=1}^{n} \lambda_i \geq \left(\prod_{i=1}^{n} \lambda_i \right)^{1/n} = (\det(TST^\star))^{1/n} = (\det S)^{1/n} .$$

To find the equality case in (2.4) take the orthogonal matrix U such that $S = U^\star D U$, where D is diagonal. Let

$$T = \left(\frac{D}{(\det S)^{1/n}} \right)^{-\frac{1}{2}} U = D_1 U.$$

Clearly, $\det T = 1$. We also have

$$\frac{(\mathrm{tr}\ TST^\star)}{n} = \frac{(\mathrm{tr}\ D_1USU^\star D_1)}{n} = \frac{(\mathrm{tr}\ D_1^2 D)}{n} = (\det S)^{1/n}.$$

Let $S = \sum_{j=1}^m c_j\alpha_j u_j \otimes u_j$. Following our last observation we can find a matrix T with $\det T = 1$ such that $(\det S)^{1/n} = \frac{\mathrm{tr}\ (TST^\star)}{n}$. Note that

$$T(u_j \otimes u_j)T^\star = Tu_j u_j^\star T^\star = Tu_j(Tu_j)^\star = (Tu_j) \otimes (Tu_j).$$

Therefore

$$\begin{aligned}\left(\det\left(\sum_{j=1}^m c_j\alpha_j u_j \otimes u_j\right)\right)^{1/n} &= \frac{1}{n}\mathrm{tr}\ \left(\sum_{j=1}^m c_j\alpha_j Tu_j \otimes Tu_j\right) \\ &= \frac{1}{n}\sum_{j=1}^m c_j\alpha_j |Tu_j|_2^2 \geq \prod_{j=1}^m \left(\alpha_j |Tu_j|_2^2\right)^{\frac{c_j}{n}} \geq \prod_{j=1}^m \alpha_j^{\frac{c_j}{n}}.\end{aligned}$$

The second inequality follows from point (2) of our lemma. □

Besides the lemma, we need the notion of mass transportation. Let us now briefly introduce it.

2.6 Definition. *Let μ be a finite Borel measure on $\mathbb{R}^d$ and let $T : \mathbb{R}^d \to \mathbb{R}^d$ be measurable. The pushforward of μ by T is a measure T_μ on $\mathbb{R}^d$ defined by*

$$T_\mu(A) = \mu(T^{-1}(A)), \quad A \in \mathcal{B}(\mathbb{R}^d).$$

If $\nu = T_\mu$ then we say that T transports μ onto ν.

Note that if $\nu = T_\mu$ then for all bounded Borel functions $h : \mathbb{R}^d \to \mathbb{R}$ we have

$$\int_{\mathbb{R}^d} h(y)\ \mathrm{d}\nu(y) = \int_{\mathbb{R}^d} h(T(x))\ \mathrm{d}\mu(x).$$

If μ and ν are absolutely continuous with respect to the Lebesgue measure, i.e. $\mathrm{d}\mu(x) = f(x)\mathrm{d}x$ and $\mathrm{d}\nu(y) = g(y)\mathrm{d}y$ then

$$\int_{\mathbb{R}^d} h(y)g(y)\ \mathrm{d}y = \int_{\mathbb{R}^d} h(T(x))f(x)\ \mathrm{d}x.$$

Assuming T is C^1 on $\mathbb{R}^d$, we obtain by changing the variable in the first integral

$$\int_{\mathbb{R}^d} h(y)g(y)\ \mathrm{d}y = \int_{\mathbb{R}^d} h(T(x))g(T(x))|\det dT(x)|\ \mathrm{d}x,$$

where dT is the differential of T. Therefore μ almost everywhere we have

$$g(T(x))|\det dT(x)| = f(x).$$

This is the so called transport equation (or a Monge-Ampère equation). Assume that μ and ν are probabilistic measures absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, say measures with densities $f, g \geq 0$. There exists a map $T : \mathbb{R} \to \mathbb{R}$ which is non-decreasing and which transports μ onto ν. Indeed, define T by

$$\int_{-\infty}^{x} f(t)\ \mathrm{d}t = \int_{-\infty}^{T(x)} g(u)\ \mathrm{d}u.$$

If

$$R(x) = \int_{-\infty}^{x} g(t)\ \mathrm{d}t$$

then

$$T(x) = R^{-1}\left(\int_{-\infty}^{x} f(t)\ \mathrm{d}t\right).$$

The simplest case is when f and g are continuous and strictly positive. Then T is of class C^1 and

$$T'(x)g(T(x)) = f(x), \qquad x \in \mathbb{R}.$$

In higher dimensions for T we can set the so called Knöthe map [61] or Brenier map [28]. For instance, the Brenier map is of the form $T = \nabla\phi$, where ϕ is a convex function.

Proof of Theorem 2.2. We have $\mathrm{Id} = \sum_{j=1}^{m} c_j u_j \otimes u_j$ and $|u_j|_2^2 = 1$. We would like to prove

$$\int_{\mathbb{R}^n} \prod_{j=1}^{m} (f_j(\langle x, u_j, \rangle))^{c_j}\ \mathrm{d}x \leq \prod_{j=1}^{m} \left(\int f_j\right)^{c_j}.$$

By homogeneity we can assume that $\int f_j = 1$. Moreover, let us suppose that each f_j is continuous and strictly positive. Let $g(s) = e^{-\pi s^2}$. Then $\int g = 1$. Let $T_j : \mathbb{R} \to \mathbb{R}$ be the map which transports $f_j(x)\mathrm{d}x$ onto $g(s)\mathrm{d}s$, i.e.

$$\int_{-\infty}^{t} f_j(s)\ \mathrm{d}s = \int_{-\infty}^{T_j(t)} g(s)\ \mathrm{d}s.$$

We have the transport equation $f_j(t) = T_j'(t)g(T_j(t))$. Hence, using (3) of Lemma 2.5 we obtain

$$\int_{\mathbb{R}^n} \prod_{j=1}^{m} (f_j(\langle x, u_j \rangle))^{c_j}\ \mathrm{d}x = \int_{\mathbb{R}^n} \prod_{j=1}^{m} \left(T_j'(\langle x, u_j \rangle)\right)^{c_j} \prod_{j=1}^{m} (g(T_j(\langle x, u_j \rangle)))^{c_j}\ \mathrm{d}x$$

$$\leq \int_{\mathbb{R}^n} \det\left(\sum_{j=1}^m c_j T_j'\left(\langle x, u_j\rangle\right) u_j \otimes u_j\right) \exp\left(-\pi \sum_{j=1}^m c_j \left(T_j(\langle x, u_j\rangle)\right)^2\right) \mathrm{d}x.$$

Note that $T_j' > 0$ since f and g are strictly positive and continuous. Let

$$y = \sum_{j=1}^m c_j T_j\left(\langle x, u_j\rangle\right) u_j.$$

Note that

$$\frac{\partial y}{\partial x_i} = \sum_{j=1}^m c_j T_j'\left(\langle x, u_j\rangle\right) \langle u_j, e_i\rangle\, u_j$$

and therefore

$$\mathrm{D}_y(x) = \sum_{j=1}^m c_j T_j'\left(\langle x, u_j\rangle\right) u_j \otimes u_j.$$

By (1) of Lemma 2.5 we have

$$\sum_{j=1}^m c_j \left(T_j\left(\langle x, u_j\rangle\right)\right)^2 \geq |y|_2^2,$$

thus, changing variables we arrive at

$$\int_{\mathbb{R}^n} \prod_{j=1}^m \left(f_j(\langle x, u_j, \rangle)\right)^{c_j} \mathrm{d}x \leq \int_{\mathbb{R}^n} \exp\left(-\pi|y|_2^2\right) \mathrm{d}y = 1.$$

For general integrable functions $f_j : \mathbb{R} \to \mathbb{R}^+$, let $\varepsilon > 0$ and define $f_j^{(\varepsilon)} = f_j \star g_\varepsilon$ where g_ε is a centered Gaussian variable of variance ε^2. The new function $f_j^{(\varepsilon)}$ is C^1 and strictly positive so the inequality holds true for the functions $(f_1^{(\varepsilon)}, \ldots, f_m^{(\varepsilon)})$. Letting $\varepsilon \to 0$, the classical Fatou lemma gives the inequality for $(f_1, \ldots, f_m)$. □

2.3 Consequences of the Brascamp-Lieb inequality

Let us state the reverse isoperimetric inequality.

2.7 Theorem. *Let K be a symmetric convex body in $\mathbb{R}^n$. Then there exists an affine transformation $\widetilde{K}$ of K such that*

$$|\widetilde{K}| = |B_\infty^n|, \quad \textit{and} \quad |\partial \widetilde{K}| \leq |\partial B_\infty^n| \tag{2.5}$$

or equivalently

$$\frac{|\partial K|}{|K|^{\frac{n-1}{n}}} \leq \frac{|\partial B_\infty^n|}{|B_\infty^n|^{\frac{n-1}{n}}} = 2n. \tag{2.6}$$

Before we give a proof of Theorem 2.7 we introduce the notion of the volume ratio.

2.8 Definition. *Let $K \subset \mathbb{R}^n$ be a convex body. The volume ratio of K is defined as*

$$vr(K) = \inf\left\{ \left(\frac{|K|}{|\mathcal{E}|}\right)^{1/n}, \quad \mathcal{E} \subset K \ \textit{ is an ellipsoid} \right\}.$$

The ellipsoid of maximal volume contained in K is called the John ellipsoid. If the John ellipsoid of K is equal to B_2^n then we say that K is in the John position.

We have the following two theorems.

2.9 Theorem. *For every symmetric convex body $K \subset \mathbb{R}^n$ we have*

$$vr(K) \le vr(B_\infty^n) = \frac{2}{(|B_2^n|)^{1/n}}. \tag{2.7}$$

2.10 Theorem. *If $B_2^n \subset K$ is the ellipsoid of maximal volume contained in a symmetric convex body $K \subset \mathbb{R}^n$ then there exist $c_1, \dots, c_m > 0$ and contact points $u_1, \dots, u_m \in \mathbb{R}^n$ such that $|u_j|_2 = \|u_j\|_K = \|u_j\|_{K^\circ} = 1$ for $1 \le j \le m$ and*

$$\mathrm{Id}_{\mathbb{R}^n} = \sum_{j=1}^m c_j u_j \otimes u_j. \tag{2.8}$$

Here we do not give a proof of Theorem 2.10. Originally, John [58] proved it with a simple extension of the Karush, Kuhn and Tucker theorem in optimisation to a compact set of constraints (instead of finite number of constraints). We refer to [52] for a modern presentation, very close to the original approach of John. We only show how John's theorem implies Theorem 2.9.

Proof of Theorem 2.9. The quantity $vr(K)$ is invariant under invertible linear transformations.We let as an exercise to check that the ellipsoid of maximal volume contained in K is unique. Therefore we may assume that the John ellipsoid of K is B_2^n. Using Theorem 2.10 we find numbers $c_1, \dots, c_m > 0$ and unit vectors $u_1, \dots, u_m \in \mathbb{R}^n$ on the boundary of K such that

$$\mathrm{Id}_{\mathbb{R}^n} = \sum_{j=1}^m c_j u_j \otimes u_j.$$

Since $u_j \in \partial B_2^n \cap \partial K$ and K is symmetric we get

$$K \subset K' := \{x \in \mathbb{R}^n, \ \ |\langle x, u_j\rangle| \le 1, \quad \text{for all } \ 1 \le j \le m\}.$$

Let $f_j(t) = \mathbf{1}_{[-1,1]}(t)$ for $1 \le j \le m$. Note that $f_j = f_j^{c_j}$, $1 \le j \le m$. From Theorem 2.2 we have

$$|K| \le |K'| = \int_{\mathbb{R}^n} \prod_{j=1}^m f_j^{c_j}\left(\langle x, u_j\rangle\right) \mathrm{d}x \le \prod_{j=1}^m \left(\int f_j\right)^{c_j} = 2^{\sum_{j=1}^m c_j} = 2^n = |B_\infty^n|.$$

Clearly, this also yields that B_2^n is the John ellipsoid for the cube B_∞^n. Therefore

$$vr(B_\infty^n) = \frac{2}{(|B_2^n|)^{1/n}}.$$

□

We finish our considerations on the reverse isoperimetric problem showing that Theorem 2.9 implies Theorem 2.7.

Proof of Theorem 2.7. Let $\widetilde{K}$ be the linear image of K such that $B_2^n \subset \widetilde{K}$ is the John ellipsoid of $\widetilde{K}$. By Theorem 2.9 we have $|\widetilde{K}| \leq 2^n$. Hence,

$$\begin{aligned} |\partial\widetilde{K}| &= \liminf_{\varepsilon\to 0^+} \frac{|\widetilde{K}+\varepsilon B_2^n| - |\widetilde{K}|}{\varepsilon} \leq \liminf_{\varepsilon\to 0^+} \frac{|\widetilde{K}+\varepsilon \widetilde{K}| - |\widetilde{K}|}{\varepsilon} \\ &= n|\widetilde{K}| = n|\widetilde{K}|^{\frac{n-1}{n}} \cdot |\widetilde{K}|^{\frac{1}{n}} \leq 2n|\widetilde{K}|^{\frac{n-1}{n}}. \end{aligned}$$

This finishes the proof as the ratio $\frac{|\partial K|}{|K|^{\frac{n-1}{n}}}$ is affine invariant. □

We state yet another application of the Brascamp-Lieb inequality.

2.11 Theorem. *If K is a symmetric convex body in the John position then $\mathbb{E}\,\|G\|_K \geq \mathbb{E}|G|_\infty$, where G is the standard Gaussian vector in $\mathbb{R}^n$, i.e. the vector $(g_1, \ldots, g_n)$ where $(g_i)_{i\leq n}$ are independent standard Gaussian random variables.*

Proof. As in the proof of Theorem 2.7 we consider numbers $c_1, \ldots, c_m > 0$ and vectors $u_1, \ldots, u_m$ satisfying the assertion of the Theorem 2.10. Note that

$$K \subset K' = \{x \in \mathbb{R}^n, \ |\langle x, u_j\rangle| \leq 1 \quad 1 \leq j \leq m\}.$$

Clearly,

$$\|G\|_K \geq \|G\|_{K'} = \max_{1\leq j\leq m} |\langle G, u_j\rangle|.$$

Moreover,

$$\mathbb{E}\,\|G\|_{K'} = \int_0^{+\infty} \mathbb{P}\left(\max_j |\langle G, u_j\rangle| \geq t\right) \mathrm{d}t.$$

We have $|G|_\infty = \max_{1\leq j\leq m} |\langle G, e_j\rangle|$ so that

$$\mathbb{E}|G|_\infty = \int_0^{+\infty} \mathbb{P}\left(\max_j |\langle G, e_j\rangle| \geq t\right) \mathrm{d}t = \int_0^{+\infty} \left(1 - \mathbb{P}\left(|g| \leq t\right)^n\right) \mathrm{d}t,$$

where g is the standard Gaussian random variable. To get the conclusion, it suffices to prove

$$\mathbb{P}\left(\max_j |\langle G, u_j\rangle| \leq t\right) \leq \left(\mathbb{P}\left(|g| \leq t\right)\right)^n.$$

Take

$$h_j(s) = \mathbf{1}_{[-t,t]}(s)\frac{e^{-s^2/2}}{\sqrt{2\pi}}, \quad f_j(s) = \mathbf{1}_{[-t,t]}(s).$$

Since

$$|x|_2^2 = \sum_{j=1}^m c_j \langle x, u_j \rangle^2 ,$$

Theorem 2.2 implies that

$$\begin{aligned}
\mathbb{P}\left(\max_j |\langle G, u_j\rangle| \le t\right) &= \int_{\mathbb{R}^n} \mathbf{1}_{\{(\max_j |\langle x,u_j\rangle|)\le t\}} \frac{1}{(2\pi)^{n/2}} e^{-|x|_2^2/2} \, \mathrm{d}x \\
&= \int_{\mathbb{R}^n} \prod_{j=1}^m f_j^{c_j}\left(\langle x, u_j\rangle\right) \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{|\langle x, u_j\rangle|^2}{2}\right)^{c_j} \mathrm{d}x \\
&= \int_{\mathbb{R}^n} \prod_{j=1}^m h_j\left(\langle x, u_j\rangle\right)^{c_j} \mathrm{d}x \\
&\le \prod_{j=1}^m \left(\int h_j\right)^{c_j} = \left(\int_{-t}^t \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \, \mathrm{d}u\right)^n \\
&= \left(\mathbb{P}\left(|g| \le t\right)\right)^n ,
\end{aligned}$$

where we have used the fact that $\sum_{j=1}^m c_j = n$. □

2.4 Notes and comments

This section is devoted to the study of the Brascamp-Lieb inequalities [26] in a convex geometric setting. As we emphasized, this approach is due to Ball [7] where he proved Theorem 2.9 and Theorem 2.7. We refer to [9] for a large survey on this subject. The proof using mass transportation approach is taken from [12]. It is important to notice a significant development of this study, the reverse Brascamp-Lieb inequality due to Barthe [11]. Theorem 2.11 is due to Schechtman and Schmuckenschläger [86] and has a very nice application in the study of Dvoretzky's theorem, because it gives a Euclidean structure associated with a convex body where the minimum among convex bodies K of $M(K) = \int_{S^{n-1}} \|x\| d\sigma_n(x)$ is known and attained for the cube. We refer to Section 4 to learn about it. A non-symmetric version of these results is known, see [7, 88, 10].

3 Borell and Prékopa-Leindler type inequalities, the notion of Ball's bodies

3.1 Brunn-Minkowski inequality

Brunn discovered the following important theorem about sections of a convex body.

3.1 Theorem. *Let $n \geq 2$ and let K be a convex body in $\mathbb{R}^n$. Take $\theta \in S^{n-1}$ and define*

$$H_r = \{x \in \mathbb{R}^n,\ \langle x, \theta \rangle = r\} = r\theta + \theta^\perp.$$

Then the function

$$r \mapsto (\mathrm{vol}(H_r \cap K))^{\frac{1}{n-1}}$$

is concave on its support.

Minkowski restated this result providing a powerful tool.

3.2 Theorem. *If A and B are non-empty compact sets then for all $\lambda \in [0,1]$ we have*

$$\mathrm{vol}\left((1-\lambda)A + \lambda B\right)^{1/n} \geq (1-\lambda)(\mathrm{vol}\, A)^{1/n} + \lambda(\mathrm{vol}\, B)^{1/n}. \tag{3.1}$$

Note that if either $A = \varnothing$ or $B = \varnothing$, this inequality does not hold in general since $(1-\lambda)A + \lambda B = \varnothing$. We can use homogeneity of volume to rewrite Brunn-Minkowski inequality in the form

$$\mathrm{vol}\,(A+B)^{1/n} \geq (\mathrm{vol}\, A)^{1/n} + (\mathrm{vol}\, B)^{1/n}. \tag{3.2}$$

At this stage, there is always a discussion between people who prefer to state the Brunn-Minkowski inequality for Borel sets (but it remains to prove that if A and B are Borel sets then $A+B$ is a measurable set) and people who prefer to work with approximation and say that for any measurable set C, $\mathrm{vol}\, C$ is the supremum of the volume of the compact sets contained in C. We choose the second way in this presentation.

The proof of the theorem of Brunn follows easily. For any $t \in \mathbb{R}$, define $A_t = \{x \in \theta^\perp, x + t\theta \in K\}$. Observe that when $s = (1-\lambda)r + \lambda t$, only the inclusion

$$A_s \supset \lambda A_t + (1-\lambda)A_r$$

is important. And inequality (3.1) applied in $\theta^\perp$ which is of dimension $n-1$ leads to the conclusion.

We can also deduce from inequality (3.2) the isoperimetric inequality.

3.3 Theorem. *Among sets with prescribed volume, the Euclidean balls are the one with minimum surface area.*

Proof. By a compact approximation of C, we can assume that C is compact and $\operatorname{vol} C = \operatorname{vol} B_2^n$. We have

$$\operatorname{vol} \partial C = \liminf_{\varepsilon \to 0+} \frac{\operatorname{vol}(C + \varepsilon B_2^n) - \operatorname{vol}(C)}{\varepsilon}.$$

By the Brunn-Minkowski inequality (3.1), we get

$$\operatorname{vol}(C + \varepsilon B_2^n)^{1/n} \geq (\operatorname{vol} C)^{1/n} + \varepsilon (\operatorname{vol} B_2^n)^{1/n},$$

hence

$$\operatorname{vol}(C + \varepsilon B_2^n) \geq (1+\varepsilon)^n \operatorname{vol} C,$$

so

$$\operatorname{vol}(\partial C) \geq \liminf_{\varepsilon \to 0+} \frac{((1+\varepsilon)^n - 1) \operatorname{vol}(C)}{\varepsilon} = n \operatorname{vol}(C) = n \operatorname{vol}(B_2^n) = \operatorname{vol}(\partial B_2^n).$$

□

There is an a priori weaker statement of the Brunn-Minkowski inequality. Applying the AM–GM inequality to the right hand side of (3.1) we get

$$|(1-\lambda)A + \lambda B| \geq |A|^{1-\lambda} |B|^{\lambda}, \qquad \lambda \in [0,1]. \tag{3.3}$$

Note that this inequality is valid for any compact sets A and B (the assumption that A and B are non-empty is no longer needed). We can see that there is no appearance of dimension in this expression.

The strong version of the Brunn-Minkowski inequality (3.1) tells us that the Lebesgue measure is a $\frac{1}{n}$-concave measure. The weaker statement (3.3) justifies that it is a log-concave measure.

3.4 Definition. *A measure μ on $\mathbb{R}^n$ is log-concave if for all compact sets A and B we have*

$$\mu((1-\lambda)A + \lambda B) \geq \mu(A)^{1-\lambda} \mu(B)^{\lambda}, \qquad \lambda \in [0,1].$$

3.5 Definition. *The function $f : \mathbb{R}^n \to \mathbb{R}$ is log-concave if for all $x, y \in \mathbb{R}^n$ we have*

$$f((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} f(y)^{\lambda}, \qquad \lambda \in [0,1].$$

Note that these definitions are dimension free.

The weak form of the inequality (3.3) for the Lebesgue measure is in fact equivalent to the strong inequality (3.1). It is a consequence of the homogeneity of the Lebesgue measure. Indeed, if

$$\mu = \frac{\lambda (\operatorname{vol} B)^{1/n}}{(1-\lambda)(\operatorname{vol} A)^{1/n} + \lambda (\operatorname{vol} B)^{1/n}}$$

then

$$\operatorname{vol}\left(\frac{(1-\lambda)A+\lambda B}{(1-\lambda)(\operatorname{vol}A)^{1/n}+\lambda(\operatorname{vol}B)^{1/n}}\right)=\operatorname{vol}\left((1-\mu)\frac{A}{(\operatorname{vol}A)^{1/n}}+\mu\frac{B}{(\operatorname{vol}B)^{1/n}}\right)$$
$$\geq \operatorname{vol}\left(\frac{A}{(\operatorname{vol}A)^{1/n}}\right)^{1-\mu}\left(\frac{B}{(\operatorname{vol}B)^{1/n}}\right)^{\mu}=1.$$

3.2 Functional version of the Brunn-Minkowski inequality

If we take $f=\mathbf{1}_A$, $g=\mathbf{1}_B$ and $m=\mathbf{1}_{(1-\lambda)A+\lambda B}$ then (3.3) says that

$$\int m \geq \left(\int f\right)^{1-\lambda}\left(\int g\right)^{\lambda}$$

and obviously m, f, g satisfies

$$m((1-\lambda)x+\lambda y)\geq f(x)^{1-\lambda}g(y)^{\lambda}.$$

We will prove the following functional version of the Brunn-Minkowski inequality called the Prékopa-Leindler inequality. This will conclude the proof of inequality (3.1) and of Theorem 3.1.

3.6 Theorem. *Let f, g, m be nonnegative measurable functions on $\mathbb{R}^n$ and let $\lambda \in [0,1]$. If for all $x, y \in \mathbb{R}^n$ we have*

$$m((1-\lambda)x+\lambda y)\geq f(x)^{1-\lambda}g(y)^{\lambda},$$

then

$$\int_{\mathbb{R}^n} m \geq \left(\int_{\mathbb{R}^n} f\right)^{1-\lambda}\left(\int_{\mathbb{R}^n} g\right)^{\lambda}. \tag{3.4}$$

We start with proving inequalities (3.1) and (3.3) in dimension 1.

3.7 Lemma. *Let A, B be non-empty compact sets in $\mathbb{R}$. Then*

$$|(1-\lambda)A+\lambda B|\geq|(1-\lambda)A|+|\lambda B|, \quad \lambda\in[0,1].$$

Moreover, for any compact sets A, B in $\mathbb{R}$,

$$|(1-\lambda)A+\lambda B|\geq|A|^{1-\lambda}|B^{\lambda}|, \quad \lambda\in[0,1].$$

Proof. Observe that the operations $A\to A+v_1$, $B\to B+v_2$ where $v_1, v_2\in\mathbb{R}$ do not change the volumes of A, B and $(1-\lambda)A+\lambda B$ (adding a number to one of the sets only shifts all of this sets). Therefore we can assume that $\sup A = \inf B = 0$. But then, since $0 \in A$ and $0 \in B$, we have

$$(1-\lambda)A + \lambda B \supset (1-\lambda)A \cup (\lambda B).$$

But $(1-\lambda)A$ and (λB) are disjoint, up to the one point 0. Therefore

$$|(1-\lambda)A + \lambda B| \geq |(1-\lambda)A| + |\lambda B|,$$

hence we have proved (3.1) in dimension 1.

The log-concavity of the Lebesgue measure on $\mathbb{R}$ follows from the AM–GM inequality. □

Proof of Theorem 3.6. Step 1. Let us now justify the Prékopa-Leindler inequality in dimension 1. We can assume, considering $f\mathbf{1}_{f\leq M}$ and $g\mathbf{1}_{g\leq M}$ instead of f and g, that f, g are bounded. Note also that this inequality possesses some homogeneity. Indeed, if we multiply f, g, m by numbers c_f, c_g, c_m satisfying

$$c_m = c_f^{1-\lambda} c_g^{\lambda},$$

then the hypothesis and the assertion do not change. Therefore, taking $c_f = \|f\|_\infty^{-1}$, $c_g = \|g\|_\infty^{-1}$ and $c_m = \|f\|_\infty^{-(1-\lambda)} \|g\|_\infty^{-\lambda}$ we can assume (since we are in the situation when f and g are bounded) that $\|f\|_\infty = \|g\|_\infty = 1$. But then

$$\int_{\mathbb{R}} m = \int_0^{+\infty} |\{m \geq s\}| \, \mathrm{d}s,$$

$$\int_{\mathbb{R}} f = \int_0^1 |\{f \geq r\}| \, \mathrm{d}r,$$

$$\int_{\mathbb{R}} g = \int_0^1 |\{g \geq r\}| \, \mathrm{d}r.$$

Note also that if $x \in \{f \geq r\}$ and $y \in \{g \geq r\}$ then by the assumption of the theorem we have $(1-\lambda)x + \lambda y \in \{m \geq r\}$. Hence,

$$(1-\lambda)\{f \geq r\} + \lambda\{g \geq r\} \subset \{m \geq r\}.$$

Moreover, the sets $\{f \geq r\}$ and $\{g \geq r\}$ are non-empty for $r \in [0,1)$. This is very important since we want to use the 1-dimensional Brunn-Minkowski inequality proved in Lemma 3.7! For any non empty compact subsets $A \subset \{f \geq r\}$ and $B \subset \{g \geq r\}$, we have by Lemma 3.7, $|\{m \geq r\}| \geq (1-\lambda)|A| + \lambda|B|$. Since Lebesgue measure is inner regular, we get that

$$|\{m \geq r\}| \geq (1-\lambda)|\{f \geq r\}| + \lambda|\{g \geq r\}|.$$

Therefore, we have

$$\int m = \int_0^{+\infty} |\{m \geq r\}| \, \mathrm{d}r \geq \int_0^1 |\{m \geq r\}| \, \mathrm{d}r \geq \int_0^1 |(1-\lambda)\{f \geq r\} + \lambda\{g \geq r\}| \, \mathrm{d}r$$
$$\geq (1-\lambda) \int_0^1 |\{f \geq r\}| \, \mathrm{d}r + \lambda \int_0^1 |\{g \geq r\}| \, \mathrm{d}r = (1-\lambda) \int f + \lambda \int g$$
$$\geq \left(\int f \right)^{1-\lambda} \left(\int g \right)^{\lambda}.$$

Observe that we have actually proved a stronger inequality

$$\int m \geq (1-\lambda) \int f + \lambda \int g,$$

but under the assumption $\|f\|_\infty = \|g\|_\infty = 1$, without which the inequality does not hold as it lacks homogeneity, in contrast to (3.6).

Step 2 (the inductive step). Suppose our inequality is true in dimension $n-1$. We will prove it in dimension n.

Suppose we have numbers $y_0, y_1, y_2 \in \mathbb{R}$ satisfying $y_0 = (1-\lambda)y_1 + \lambda y_2$. Define $m_{y_0}, f_{y_1}, g_{y_2} : \mathbb{R}^{n-1} \to \mathbb{R}_+$ by

$$m_{y_0}(x) = m(y_0, x), \quad f_{y_1}(x) = f(y_1, x), \quad g_{y_2}(x) = (y_2, x),$$

where $x \in \mathbb{R}^{n-1}$. Note that since $y_0 = (1-\lambda)y_1 + \lambda y_2$ we have

$$m_{y_0}((1-\lambda)x_1 + \lambda x_2) = m((1-\lambda)y_1 + \lambda y_2, (1-\lambda)x_1 + \lambda x_2)$$
$$\geq f(y_1, x_1)^{1-\lambda} g(y_2, x_2)^{\lambda} = f_{y_1}(x_1)^{1-\lambda} g_{y_2}(x_2)^{\lambda},$$

hence m_{y_0}, f_{y_1} and g_{y_2} satisfy the assumption of the $(n-1)$-dimensional Prékopa-Leindler inequality. Therefore we have

$$\int_{\mathbb{R}^{n-1}} m_{y_0} \geq \left(\int_{\mathbb{R}^{n-1}} f_{y_1} \right)^{1-\lambda} \left(\int_{\mathbb{R}^{n-1}} g_{y_2} \right)^{\lambda}.$$

Define new functions $M, F, G : \mathbb{R} \to \mathbb{R}_+$

$$M(y_0) = \int_{\mathbb{R}^{n-1}} m_{y_0}, \quad F(y_1) = \int_{\mathbb{R}^{n-1}} f_{y_1}, \quad G(y_2) = \int_{\mathbb{R}^{n-1}} g_{y_2}.$$

We have seen (the above inequality) that when $y_0 = (1-\lambda)y_1 + \lambda y_2$ then there holds

$$M((1-\lambda)y_1 + \lambda y_2) \geq F(y_1)^{1-\lambda} G(y_2)^{\lambda}.$$

Hence, by 1-dimensional Prékopa-Leindler inequality proved in Step 1, we get

$$\int_{\mathbb{R}} M \geq \left(\int_{\mathbb{R}} F\right)^{1-\lambda} \left(\int_{\mathbb{R}} G\right)^{\lambda}.$$

But

$$\int_{\mathbb{R}} M = \int_{\mathbb{R}^n} m, \quad \int_{\mathbb{R}} F = \int_{\mathbb{R}^n} f, \quad \int_{\mathbb{R}} G = \int_{\mathbb{R}^n} g,$$

so we conclude that

$$\int_{\mathbb{R}^n} m \geq \left(\int_{\mathbb{R}^n} f\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g\right)^{\lambda}.$$

□

The next theorem will be useful in the sequel to prove the functional version of the so-called Blaschke-Santaló inequality, Theorem 3.11.

3.8 Theorem. *Suppose* $f, g, m : [0,\infty) \to [0,\infty)$ *are measurable and suppose there exists* $\lambda \in [0,1]$ *such that*

$$m(t) \geq f(r)^{1-\lambda} g(s)^{\lambda}, \quad \textit{whenever } t = r^{1-\lambda} s^{\lambda}$$

Then

$$\int m \geq \left(\int f\right)^{1-\lambda} \left(\int g\right)^{\lambda}. \tag{3.5}$$

Proof. This inequality has a lot of homogeneity. Again, if we multiply f, g, m by numbers c_f, c_g, c_m satisfying

$$c_m = c_f^{1-\lambda} c_g^{\lambda},$$

then the hypothesis and the assertion do not change. Moreover, we can rescale arguments of f, g, m by d_f, d_g, d_m in such a way that

$$d_m = d_f^{1-\lambda} d_g^{\lambda}.$$

We can assume, by taking $f\mathbf{1}_{f\leq M}\mathbf{1}_{[-M,M]}$, $g\mathbf{1}_{g\leq M}\mathbf{1}_{[-M,M]}$ that f and g are bounded and have compact support. Moreover, by scaling we can assume that

$$\sup rf(r) = \sup rg(r) = 1. \tag{3.6}$$

Let

$$M(x) = e^x m(e^x), \quad F(x) = e^x f(e^x), \quad G(x) = e^x g(e^x).$$

Clearly, changing variables we have

$$\int_0^{+\infty} m(t)\,\mathrm{d}t = \int_{-\infty}^{+\infty} M(\omega)\,\mathrm{d}\omega, \quad \int_0^{+\infty} f(t)\,\mathrm{d}t = \int_{-\infty}^{+\infty} F(\omega)\,\mathrm{d}\omega,$$

$$\int_0^{+\infty} g(t)\,\mathrm{d}t = \int_{-\infty}^{+\infty} G(\omega)\,\mathrm{d}\omega.$$

By (3.6), we get

$$\int_{-\infty}^{+\infty} F(\omega)\,\mathrm{d}\omega = \int_0^1 |\{F \geq r\}|\,\mathrm{d}r \quad \text{and} \quad \int_{-\infty}^{+\infty} G(\omega)\,\mathrm{d}\omega = \int_0^1 |\{G \geq r\}|\,\mathrm{d}r.$$

By the hypothesis of f, g and m we have

$$\begin{aligned} M((1-\lambda)u + \lambda v) &= m((e^u)^{1-\lambda}(e^v)^{\lambda})(e^u)^{1-\lambda}(e^v)^{\lambda} \\ &\geq (f(e^u))^{1-\lambda}(g(e^v))^{\lambda}(e^u)^{1-\lambda}(e^v)^{\lambda} = (F(u))^{1-\lambda}(G(v))^{\lambda}. \end{aligned} \tag{3.7}$$

Hence, for any $r \in [0,1)$, if $x \in \{F \geq r\}$ and $y \in \{G \geq r\}$, then $(1-\lambda)x+\lambda y \in \{M \geq r\}$. The sets $\{F \geq r\}$ and $\{G \geq r\}$ are not empty therefore by Lemma 3.7 (which is the 1-dimensional Brunn-Minkowski inequality), for any non empty compact sets $A \subset \{F \geq r\}$ and $B \subset \{G \geq r\}$, $|\{M \geq r\}| \geq (1-\lambda)|A| + \lambda|B|$. Since Lebesgue measure is inner regular, we conclude that $|\{M \geq r\}| \geq (1-\lambda)|\{F \geq r\}| + \lambda|\{G \geq r\}|$ and

$$\begin{aligned} \int_{-\infty}^{+\infty} M(\omega)\,\mathrm{d}\omega \geq \int_0^1 |\{M \geq r\}|\,\mathrm{d}r &\geq (1-\lambda)\int_{-\infty}^{+\infty} F(\omega)\,\mathrm{d}\omega + \lambda\int_{-\infty}^{+\infty} G(\omega)\,\mathrm{d}\omega \\ &\geq \left(\int_{-\infty}^{+\infty} F(\omega)\,\mathrm{d}\omega\right)^{1-\lambda}\left(\int_{-\infty}^{+\infty} G(\omega)\,\mathrm{d}\omega\right)^{\lambda}. \end{aligned}$$

□

Note that after establishing (3.7) we could have directly used the 1-dimensional Prékopa-Leindler inequality, Theorem 3.6. But we can also recover Theorem 3.6. Indeed, let

$$M(t) = |\{m \geq t\}|, \quad F(r) = |\{f \geq r\}|, \quad G(s) = |\{g \geq s\}|.$$

We have to prove that

$$\int_0^{+\infty} M(t)\,\mathrm{d}t \geq \left(\int_0^{+\infty} F(r)\,\mathrm{d}r\right)^{1-\lambda}\left(\int_0^{+\infty} G(s)\,\mathrm{d}s\right)^{\lambda}.$$

Note that if $t = r^{1-\lambda}s^{\lambda}$, then from the hypothesis of Theorem 3.6 we have

$$\{m \geq t\} \supset (1-\lambda)\{f \geq r\} + \lambda\{g \geq s\}.$$

From Lemma 3.7, we get

$$M(t) \geq F(r)^{1-\lambda}G(s)^{\lambda}$$

(even if the sets are empty, because we just use the log-concavity of the Lebesgue measure on $\mathbb{R}$). We conclude by using Theorem 3.8.

3.3 Functional version of the Blaschke-Santaló inequality

We only recall the Blaschke-Santaló inequality (without the proof). We discuss the symmetric case.

3.9 Definition. *Let C be a compact and symmetric set in $\mathbb{R}^n$. We define the polar body C° by*

$$C^\circ = \{y \in \mathbb{R}^n,\ \forall x \in C \mid \langle x, y\rangle \mid \leq 1\}.$$

3.10 Theorem. *Let C be a compact and symmetric set in $\mathbb{R}^n$. Then*

$$|C| \cdot |C^\circ| \leq |B_2^n|^2 . \tag{B-S}$$

Nevertheless, we will prove its functional version using (B-S) inequality.

3.11 Theorem. *Suppose $f, g : \mathbb{R}^n \to [0, \infty)$ and $\Omega : [0, \infty) \to [0, \infty)$ are integrable and f, g are even. Suppose that $\Omega(t) \geq \sqrt{f(x)g(y)}$ whenever $|\langle x, y\rangle| \geq t^2$. Then*

$$\int_{\mathbb{R}^n} \Omega(|x|_2)\ \mathrm{d}x = n|B_2^n| \int_0^{+\infty} t^{n-1}\Omega(t)\ \mathrm{d}t \geq \left(\int f\right)^{1/2} \left(\int g\right)^{1/2} . \tag{3.8}$$

3.12 Remark. We can recover the classical version of the (B-S) inequality from the functional one. Take $f = \mathbf{1}_C$, $g = \mathbf{1}_{C^\circ}$ and $\Omega = \mathbf{1}_{[0,1]}$. If $x \in C$ and $y \in C^\circ$ then $|\langle x, y\rangle| \leq 1$. Hence, if $t > 1$ then $\Omega(t) = \sqrt{f(x)g(y)} = 0$. If $t \leq 1$ then obviously $1 = \Omega(t) \geq \sqrt{f(x)g(y)}$. By Theorem 3.11 we get (B-S).

Proof of Theorem 3.11. The first equality is just an integration in polar coordinates. It is enough to prove the statement for the function

$$t \mapsto \sup\{\sqrt{f(x)g(y)} : |\langle x, y\rangle| \geq t^2\},$$

so that we can assume Ω non-increasing. For $r, s, t \in \mathbb{R}_+$ we take

$$\phi(r) = |\{f \geq r\}|, \quad \psi(s) = |\{g \geq s\}|, \quad m(t) = |B_2^n| \cdot |\{\Omega \geq t\}|^n.$$

We claim that $m(\sqrt{rs}) \geq \sqrt{\phi(r)\psi(s)}$. Thanks to this we can apply Theorem 3.8 with $\lambda = 1/2$ and obtain

$$\int m \geq \left(\int \phi\right)^{1/2} \left(\int \psi\right)^{1/2}.$$

Thus, the proof of (3.8) will be finished since

$$\int_0^{+\infty} m(t)\ \mathrm{d}t = |B_2^n| \int_0^{+\infty} |\{\Omega \geq t\}|^n\ \mathrm{d}t = |B_2^n| \int_0^{+\infty} \int_0^{|\{\Omega \geq t\}|} nu^{n-1}\ \mathrm{d}u\ \mathrm{d}t$$
$$= |B_2^n| \int_0^{+\infty} nu^{n-1} \int_0^{+\infty} \mathbf{1}_{[0,|\{\Omega\geq t\}|]}(u)\ \mathrm{d}t\ \mathrm{d}u = |B_2^n| \int_0^{+\infty} nu^{n-1}\Omega(u)\ \mathrm{d}u.$$

Now we prove our claim. Let $C = \{f \geq r\}$ and $C^\circ = \{y,\ \forall x \in C \mid \langle x, y\rangle \mid \leq 1\}$. Let

$$\alpha^2 = \sup\{| \langle x, y\rangle |,\ f(x) \geq r, g(y) \geq s\}.$$

Using the definition of C, C° and α we get $\{y,\ g(y) \geq s\} \subset \alpha^2 C^\circ$. By the assumption on Ω, f, g we obtain $\Omega(u) \geq \sqrt{rs}$ for $u < \alpha$, hence $|\{\Omega \geq \sqrt{rs}\}| \geq \alpha$. Therefore, $m(\sqrt{rs}) \geq \alpha^n |B_2^n|$. By the (B-S) inequality we have $|C||C^\circ| \leq |B_2^n|^2$. Thus,

$$|B_2^n|^2 \geq |\{f \geq r\}| \cdot |C^\circ| \geq |\{f \geq r\}| \cdot |\{g \geq s\}| \alpha^{-2n},$$

so

$$\sqrt{\phi(r)\psi(s)} = \sqrt{|\{f \geq r\}| \cdot |\{g \geq s\}|} \leq |B_2^n| \alpha^n \leq m(\sqrt{rs}).$$

□

3.4 Borell and Ball functional inequalities

The following is another type of functional inequality, in the spirit of Theorem 3.6. We will see in the next section its role in convex geometry.

3.13 Theorem. *Suppose $f, g, m : (0, \infty) \to [0, \infty)$ are measurable and such that*

$$m(t) \geq \sup\left\{ f(r)^{\frac{s}{r+s}} g(s)^{\frac{r}{r+s}},\ \ \frac{1}{r} + \frac{1}{s} = \frac{2}{t} \right\}$$

for all $t > 0$. Then

$$2\left(\int_0^\infty m(t) t^{p-1}\,\mathrm{dt}\right)^{-\frac{1}{p}} \leq \left(\int_0^\infty f(t) t^{p-1}\,\mathrm{dt}\right)^{-\frac{1}{p}} + \left(\int_0^\infty g(t) t^{p-1}\,\mathrm{dt}\right)^{-\frac{1}{p}} \tag{3.9}$$

for every $p > 0$.

Proof. Considering $\min\{f_i, M\}\mathbf{1}_{\cdot \leq M}$ for $f_1 = f, f_2 = g, f_3 = m$ we can assume that f, g, m are bounded, compactly supported in $(0, \infty)$ and not 0 a.e. We do not have good homogeneity. Let $\theta > 0$ be such that

$$\sup r^{p+1} f(r) = \theta^{p+1} \sup r^{p+1} g(r). \tag{3.10}$$

Let

$$A = \left(\int_0^\infty f(t) t^{p-1}\,\mathrm{dt}\right)^{\frac{1}{p}}, B = \left(\int_0^\infty g(t) t^{p-1}\,\mathrm{dt}\right)^{\frac{1}{p}}, C = \left(\int_0^\infty m(t) t^{p-1}\,\mathrm{dt}\right)^{\frac{1}{p}}.$$

Define

$$F(u) = f\left(\frac{1}{u}\right)\left(\frac{1}{u}\right)^{p+1}, G(u) = g\left(\frac{1}{\theta u}\right)\left(\frac{1}{u}\right)^{p+1},$$

$$M(u)=\left(\frac{1+\theta}{2}\right)^{p+1} m\left(\frac{1}{u}\right)\left(\frac{1}{u}\right)^{p+1}.$$

Hence, changing the variables we have

$$\int_0^{+\infty} F(u)\,\mathrm{d}u = A^p, \int_0^{+\infty} G(u)\,\mathrm{d}u=(\theta B)^p, \int_0^{+\infty} M(u)\,\mathrm{d}u=\left(\frac{1+\theta}{2}\right)^{p+1} C^p.$$

We want to prove that

$$\frac{2}{C}\leq\frac{1}{A}+\frac{1}{B}.$$

Note that by virtue of equality (3.10) we have $\sup G=\sup F$.

We claim that

$$M(w)\geq \sup\{F(u)^{\frac{u}{u+\theta v}} G(v)^{\frac{\theta v}{u+\theta v}},\ u+\theta v=2w\},\quad w\in(0,\infty).$$

If $u+\theta v=2w$, then setting $r=1/u$, $s=1/(\theta v)$, $t=1/w$ we have $1/r+1/s=2/t$ and

$$\frac{s}{r+s}=\frac{\frac{1}{\theta v}}{\frac{1}{u}+\frac{1}{\theta v}}=\frac{u}{u+\theta v},$$

hence

$$F(u)^{\frac{u}{u+\theta v}} G(v)^{\frac{\theta v}{u+\theta v}} = f(r)^{\frac{s}{r+s}} g(s)^{\frac{r}{r+s}} \left(r^{\frac{s}{r+s}}(\theta s)^{\frac{r}{r+s}}\right)^{p+1}.$$

We obtain

$$r^{\frac{s}{r+s}} s^{\frac{r}{r+s}} \leq \frac{s}{r+s} r+\frac{r}{r+s}\theta s=(1+\theta)\frac{rs}{r+s}=\frac{1+\theta}{2}\frac{1}{w}.$$

Thus,

$$F(u)^{\frac{u}{u+\theta v}} G(v)^{\frac{\theta v}{u+\theta v}} \leq \left(\frac{1+\theta}{2}\right)^{p+1} m\left(\frac{1}{w}\right)\left(\frac{1}{w}\right)^{p+1}=M(w).$$

Summarizing, we have $\sup F=\sup G$ and

$$\frac{1}{2}\{F\geq\xi\}+\frac{\theta}{2}\{G\geq\xi\}\subset\{M\geq\xi\}.$$

Therefore, Lemma 3.7 (which is nothing else but Brunn-Minkowski inequality in dimension 1) yields that

$$\begin{aligned}\int M &\geq \int_0^{\sup F}|\{M\geq\xi\}|\,\mathrm{d}\xi\geq\frac{1}{2}\int_0^{\sup F}|\{F\geq\xi\}|\,\mathrm{d}\xi+\frac{\theta}{2}\int_0^{\sup G}|\{G\geq\xi\}|\,\mathrm{d}\xi\\ &=\frac{1}{2}\int F+\frac{\theta}{2}\int G.\end{aligned}$$

In terms of A, B, C we have

$$\left(\frac{1+\theta}{2}\right)^{p+1} C^p \geq \frac{1}{2}A^p + \frac{\theta}{2}(\theta B)^p,$$

hence

$$C^p \geq 2^p \frac{A^p + \theta^{p+1}B^p}{(1+\theta)^{p+1}}.$$

Define $\phi : [0, \infty) \to [0, \infty)$ by

$$\phi(\theta) = \frac{A^p + \theta^{p+1}B^p}{(1+\theta)^{p+1}}.$$

Since $\inf_{\mathbb{R}_+} \phi(\theta) = \phi(A/B)$ (calculate the derivative to see that ϕ is unimodal and $\phi'(A/B) = 0$), we get

$$C^p \geq 2^p \frac{A^p + A^{p+1}/B}{\left(1+\frac{A}{B}\right)^{p+1}} = 2^p \frac{A^p}{\left(1+\frac{A}{B}\right)^{p}} = \left(\frac{2AB}{A+B}\right)^p.$$

Now the proof is complete. □

3.5 Consequences in convex geometry

Having the Prékopa-Leindler inequality at hand we can constitute a handful of basic properties of log-concave measures. We begin with a simple observation that a measure with a log-concave density is log-concave.

3.14 Proposition. *If $h : \mathbb{R}^n \to \mathbb{R}_+$ is log-concave and $h \in L^1_{loc}$ then*

$$\mu(A) = \int_A h$$

defines a log-concave measure on $\mathbb{R}^n$.

Proof. For compact sets A, B take $m(z) = \mathbf{1}_{\lambda A+(1-\lambda)B}(z)h(z)$, $f(x) = \mathbf{1}_A(x)h(x)$, $g(y) = \mathbf{1}_B(y)h(y)$. Then from log-concavity of h and by the definition of the Minkowski sum we have $m(\lambda x + (1-\lambda)y) \geq f(x)^\lambda g(y)^{1-\lambda}$. Therefore, by the Prékopa-Leindler inequality, i.e. Theorem 3.6, we have $\int m \geq (\int f)^\lambda (\int g)^{1-\lambda}$, which is exactly the desired inequality

$$\mu(\lambda A + (1-\lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda}.$$

□

For example, the standard Gaussian measure and the standard symmetric exponential distribution on $\mathbb{R}^n$ are log-concave measures. Another key example is the following. Let μ be the uniform measure on a convex body $K \subset \mathbb{R}^n$, that is for any measurable set $A \subset \mathbb{R}^n$,

$$\mu(A) = \frac{|K \cap A|}{|K|}.$$

Since the function $x \mapsto \mathbf{1}_K(x)$ is log-concave, μ is log-concave. It follows also from the weak form of the Brunn-Minkowski inequality, see Lemma 3.7.

Now we show that marginal distributions of a log-concave density are again log-concave.

3.15 Theorem. *If $h : \mathbb{R}^{n+p} \to \mathbb{R}_+$ is a log-concave integrable function $(\mathbb{R}^n \times \mathbb{R}^p) \ni (x, y) \mapsto h(x, y)$, then the function*

$$\mathbb{R}^n \ni x \mapsto \int_{\mathbb{R}^p} h(x, y) \, \mathrm{d}y$$

is log-concave on $\mathbb{R}^n$.

Proof. We want to prove that for $x_0, x_1 \in \mathbb{R}^n$, $y \in \mathbb{R}^p$ and $\lambda \in [0, 1]$ we have

$$\int_{\mathbb{R}^p} h((1-\lambda)x_0 + \lambda x_1, y) \, \mathrm{d}y \geq \left(\int_{\mathbb{R}^p} h(x_0, y) \, \mathrm{d}y \right)^{\lambda} \left(\int_{\mathbb{R}^p} h(x_1, y) \, \mathrm{d}y \right)^{1-\lambda}.$$

Let

$$m(y) = h((1-\lambda)x_0 + \lambda x_1, y), \quad f(y) = h(x_0, y), \quad g(y) = h(x_1, y).$$

Then log-concavity of h yields

$$\begin{aligned} m((1-\lambda)y_0 + \lambda y_1) = h((1-\lambda)(x_0, y_0) + \lambda(x_1, y_1)) &\geq h(x_0, y_0)^{1-\lambda} h(x_1, y_1)^{\lambda} \\ = f(y_0)^{1-\lambda} g(y_1)^{\lambda}. \end{aligned}$$

Therefore, by the Prékopa-Leindler inequality, i.e. Theorem 3.6, we get

$$\int_{\mathbb{R}^p} m(y) \, \mathrm{d}y \geq \left(\int_{\mathbb{R}^p} h(x_0, y_0) \, \mathrm{d}y_0 \right)^{1-\lambda} \left(\int_{\mathbb{R}^p} h(x_1, y_1) \, \mathrm{d}y_1 \right)^{\lambda}.$$

□

A simple consequence is that the class of log-concave distributions is also closed with respect to convolving.

3.16 Proposition. *Let $f, g : \mathbb{R}^n \to \mathbb{R}_+$ be log-concave. Then the convolution $f \star g : x \mapsto \int_{\mathbb{R}^n} f(x-y)g(y)dx$ is also log-concave.*

Proof. Apply Theorem 3.15 to $h(x, y) = f(x - y)g(y)$. □

Due to the Brunn-Minkowski inequality, the function of the measures of sections of a convex body is not completely arbitrary.

3.17 Theorem. *Let K be a convex body in $\mathbb{R}^n$ and let E be k-dimensional subspace of $\mathbb{R}^n$. Let $F = E^\perp$. Then the function $f : F \to \mathbb{R}_+$*

$$f(y) = \mathrm{vol}_k((y + E) \cap K)$$

is $1/k$-concave on its support $P_F(K)$, namely

$$f^{1/k}(\lambda x + (1 - \lambda)y) \geq \lambda f^{1/k}(x) + (1 - \lambda)f^{1/k}(y),$$

when $f(x)f(y) > 0$.

Proof. As in the proof of Theorem 3.1, we deduce from convexity of K and Brunn-Minkowski inequality in $\mathbb{R}^k$, i.e. Theorem 3.2,

$$\begin{aligned} f^{1/k}(\lambda x + (1 - \lambda)y) &\geq \mathrm{vol}_k^{1/k}\left(\lambda(K \cap (x + E)) + (1 - \lambda)(K \cap (y + E))\right) \\ &\geq \lambda \, \mathrm{vol}_k^{1/k}(K \cap (x + E)) + (1 - \lambda)\, \mathrm{vol}_k^{1/k}(K \cap (y + E)) \\ &= \lambda f^{1/k}(x) + (1 - \lambda)f^{1/k}(y). \end{aligned}$$

□

3.18 Remark. If K is symmetric with respect to 0 then f is even and therefore f is maximal at 0. Moreover, it is known from a result of Fradelizi [40] that if K has center of mass at the origin then

$$\max_y f(y) \leq e^k f(0).$$

3.19 Remark. If K, L are convex bodies in $\mathbb{R}^n$, then the function

$$f(y) = \mathrm{vol}((y + L) \cap K)$$

is $\frac{1}{n}$-concave on its support, that is $K - L$. Moreover, Fradelizi [40] proved also that if $K - L$ has barycentre at the origin, then

$$\max_y \mathrm{vol}((y + L) \cap K) \leq e^n \, \mathrm{vol}(L \cap K) \tag{3.11}$$

Proof. It is enough to check that

$$(\lambda x + (1 - \lambda)y + L) \cap K \supset \lambda(x + L) \cap K + (1 - \lambda)(y + L) \cap K$$

and then the same argument as in Theorem 3.17 finishes the proof. Suppose we have a point $\lambda a + (1 - \lambda)b$, where $a \in (x + L) \cap K$ and $b \in (y + L) \cap K$. Then $a, b \in K$

and $a = x + a_0$, $b = y + b_0$, where $a_0, b_0 \in L$. Therefore, from convexity of K we have $\lambda a + (1-\lambda)b \in K$. Moreover,

$$\lambda a + (1-\lambda)b = \lambda x + (1-\lambda)y + \lambda a_0 + (1-\lambda)b_0 \in \lambda x + (1-\lambda)y + L$$

from convexity of L. $\square$

Our next observation concerns the measure of both sections and projections of convex bodies.

3.20 Proposition. *Let C be a convex body in $\mathbb{R}^n$ with non-empty interior. Let E be k-dimensional subspace of $\mathbb{R}^n$ and let $F = E^\perp$. Then*

$$|P_F(C)| \cdot \max_{y\in F} |C \cap (y+E)| \geq |C| \geq \frac{1}{\binom{n}{k}} |P_F(C)| \cdot \max_{y\in F} |C \cap (y+E)|. \tag{3.12}$$

Before giving a proof, we show the following corollary about two bodies, known as Rogers-Shephard inequalities.

3.21 Corollary. *Let A, B be two convex bodies in $\mathbb{R}^n$. Then*

$$2^n \left|\frac{A-B}{2}\right| \max_{x,y\in\mathbb{R}^n} |(A-x)\cap(B-y)| \geq |A|\cdot|B| \geq$$
$$\geq \frac{2^n}{\binom{2n}{n}} \left|\frac{A-B}{2}\right| \max_{x,y\in\mathbb{R}^n} |(A-x)\cap(B-y)|.$$

In particular, if $A-B$ has barycentre at the origin then up to a universal constant

$$(|A|\cdot|B|)^{1/n} \approx \left(\left|\frac{A-B}{2}\right| \cdot |A\cap B|\right)^{1/n}.$$

Moreover, if A, B are symmetric, then

$$2^n \left|\frac{A+B}{2}\right| \cdot |A\cap B| \geq |A|\cdot|B| \geq \frac{2^n}{\binom{2n}{n}} \left|\frac{A+B}{2}\right| \cdot |A\cap B|.$$

and

$$(|A|\cdot|B|)^{1/n} \approx \left(\left|\frac{A+B}{2}\right| \cdot |A\cap B|\right)^{1/n}.$$

Proof. Take $C = A\times B \subset \mathbb{R}^{2n}$ and

$$E = \{(x,y)\in\mathbb{R}^{2n},\ x = y\}.$$

Then

$$F = E^\perp = \{(x,y)\in\mathbb{R}^{2n},\ x+y = 0\}.$$

Note that

$$(x, y) = \frac{x+y}{2}(1,1) + \frac{x-y}{2}(1,-1).$$

Therefore, $P_F(x,y) = \frac{x-y}{2}(1,-1)$, hence

$$P_F(C) = \left\{ \frac{x-y}{2}(1,-1) \in \mathbb{R}^{2n} \mid x \in A, y \in B \right\}.$$

Consider the linear function $L : \mathbb{R}^n \to \mathbb{R}^{2n}$, $L(x) = (x, -x)$. Clearly, $L\left((A-B)/2\right) = P_F(C)$. Therefore,

$$|P_F(C)| = \left|\frac{A-B}{2}\right| \left(\sqrt{2}\right)^n.$$

Moreover,

$$(A \times B) \cap ((x,y) + E) = \Big[((A-x) \times (B-y)) \cap E\Big] + (x,y).$$

If we consider $R(x) = (x,x)$, $R : \mathbb{R}^n \to \mathbb{R}^{2n}$ then

$$R((A-x) \cap (B-y)) = ((A-x) \times (B-y)) \cap E.$$

Thus,

$$|C \cap ((x,y) + E)| = \left(\sqrt{2}\right)^n |(A-x) \cap (B-y)|,$$

and the conclusion follows from Proposition 3.20.

To prove the second inequality it suffices to observe that if $A - B$ has barycentre at the origin, we get from inequality (3.11) that

$$|A \cap B| \leq \max_{x,y \in \mathbb{R}^n} |(A-x) \cap (B-y)| \leq e^n |A \cap B|.$$

Moreover, if A and B are symmetric, then $A = -A$, $B = -B$ and $|(A-x) \cap (B-y)|$ is maximal when $x = y = 0$. □

Proof of Proposition 3.20. Consider the function $f : F \to \mathbb{R}_+$ given by $f(y) = |(y+E) \cap C|$. Obviously,

$$|C| = \int_{P_F(C)} f(y) \, \mathrm{d}y \leq |P_F(C)| \cdot \max_{y \in F} f(y).$$

The second estimate is more delicate. By translation we can assume that $\max_{y\in F} f(y) = f(0)$. Let $\|\cdot\|_{P_F(C)}$ be the gauge induced by $P_F(C)$ on F. If $y \in P_F(C)$ then $\|y\|_{P_F(C)} \leq 1$. Note that

$$y = (1 - \|y\|_{P_F(C)}) \cdot 0 + \|y\|_{P_F(C)} \frac{y}{\|y\|_{P_F(C)}}.$$

Since, by Theorem 3.17, f is $1/k$-concave on its support $P_F(C)$, $0 \in P_F(C)$, and $y/\|y\|_{P_F(C)} \in P_F(C)$, we have

$$f^{1/k}(y) \geq f^{1/k}(0)(1 - \|y\|_{P_F(C)}) + f^{1/k}(y/\|y\|_{P_F(C)})\,\|y\|_{P_F(C)} \geq f^{1/k}(0)(1 - \|y\|_{P_F(C)}).$$

Hence,

$$|C| = \int_{P_F(C)} f(y)\,\mathrm{d}y \geq f(0) \int_{P_F(C)} \left(1 - \|y\|_{P_F(C)}\right)^k \mathrm{d}y.$$

It is clear that for a convex body K in $\mathbb{R}^m$, by integrating with respect to the cone measure, we have

$$\int_K g(\|y\|_K)\,\mathrm{d}y = \int_0^1 \int_{\|y\|_K = t} g(t)\,\mathrm{d}y\,\mathrm{d}t = |K| \int_0^1 g(t) m t^{m-1}\,\mathrm{d}t,$$

since

$$\int_{\|y\|_K \leq t} 1\,\mathrm{d}y = t^m |K|, \qquad \int_{\|y\|_K = t} 1\,\mathrm{d}y = m t^{m-1} |K|.$$

Applying this to the convex body $P_F(C)$ which lives in dimension $n-k$, we get

$$|C| \geq f(0)|P_F(C)| \int_0^1 (n-k)(1-t)^k t^{n-k-1}\,\mathrm{d}t = \frac{f(0)|P_F(C)|}{\binom{n}{k}},$$

which was our goal since $f(0) = \max_{y \in F} f(y)$. □

Just to illustrate the usefulness of the functional inequalities from the previous section, we show a one dimensional result which does not seem to be obvious at first glance.

3.22 Proposition. *For $A, B \subset (0, \infty)$ we set*

$$H(A,B) = \left\{ \frac{2}{1/a + 1/b},\ a \in A, b \in B \right\}.$$

Then we have

$$|H(A,B)| \geq \frac{2|A| \cdot |B|}{|A| + |B|}. \tag{3.13}$$

Proof. Set $f = \mathbf{1}_A$, $g = \mathbf{1}_B$ and $m = \mathbf{1}_{H(A,B)}$ and use Theorem 3.13 with $p = 1$. □

At the end of this section we consider how to construct a convex body out of a log-concave function. It is a crucial observation following from Theorem 3.13. Let us emphasize its importance in the sequel (Section 6) where we establish basic properties of the so-called Z_p-bodies.

3.23 Theorem. *Suppose that a function $f : \mathbb{R}^n \to [0, \infty)$ is log-concave, integrable and not 0 a.e.. Then for $p > 0$*

$$\|x\| = \begin{cases} \left(\int_0^{+\infty} f(rx) r^{p-1}\, \mathrm{d}r \right)^{-1/p}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is a gauge on $\mathbb{R}^n$.

Proof. Obviously, $\|\lambda x\| = \lambda \|x\|$ if $\lambda > 0$ and $\|x\| = 0$ if and only if $x = 0$. Therefore, the main difficulty is to prove that

$$\|x + y\| \leq \|x\| + \|y\|.$$

Fix $x, y \in \mathbb{R}^n$. Let us take $g(r) = f(rx)$, $h(s) = f(sy)$ and $m(t) = f(\frac{1}{2}t(x+y))$ for $r, s, t \geq 0$. Suppose $1/r + 1/s = 2/t$. Let $\lambda = r/(r+s)$ so that $t/2 = \lambda s = (1-\lambda) r$. By log-concavity of f

$$m(t) = f\left(\frac{1}{2}t(x+y)\right) \geq f(rx)^{1-\lambda} f(sy)^{\lambda} = g(r)^{\frac{s}{r+s}} h(s)^{\frac{r}{r+s}}.$$

Now it suffices to use Theorem 3.13 for m, g and h. □

The previous theorem can be seen as a generalisation of a theorem due to Busemann from [29]. Choosing f and p suitably we obtain the following result.

3.24 Theorem. *Let K be a symmetric convex body with 0 in its interior. Then*

$$\|x\| = \frac{|x|_2}{|x^{\perp} \cap K|_2}$$

is a norm on $\mathbb{R}^n$.

3.6 Notes and comments

Most of the material of this section is taken from the PhD Thesis of Keith Ball [5]. Historically, the names of Prékopa and Leindler stay attached to Theorem 3.6. Indeed, Prékopa [81, 82, 83] studied a lot the notion of log-concave functions. Theorem 3.6 is the culmination in this theory, and yet it is a simple statement. Prékopa's proof uses an argument of transport of mass which can be traced back to Knöthe [61]. On the other hand, Borell [24] submitted his paper only six months after the paper of Prékopa and he presented a more general version of the inequality. But it seems that the general version of a Theorem of Borell [24] has been forgotten. This is why we would like to restate it here.

3.25 Theorem. *Let $\Omega_1, \dots, \Omega_N$ be open subsets of $\mathbb{R}^n$ and let $\phi : \Omega_1 \times \dots \times \Omega_N \to \mathbb{R}^n$ be a C^1 function such that*

$$\phi = (\phi_1, \dots, \phi_n) \text{ and } \frac{\partial \phi_j}{\partial x_i^k} > 0$$

for all $j \in \{1, \dots, n\}$ and all $i \in \{1, \dots, N\}, k \in \{1, \dots, n\}$. Define

$$\Omega_0 = \phi(\Omega_1, \dots, \Omega_N), \qquad \text{and } d\mu_i = f_i(x)dx \text{ where } f_i \in L_1^{loc}(\Omega_i), i = 0, 1, \dots, n.$$

Suppose $\Phi : [0, +\infty)^N \to [0, +\infty)$ is a continuous function homogeneous of degree one and increasing in each variable separately. Then the inequality

$$\mu_0(\phi(A_1, \dots, A_N)) \geq \Phi(\mu_1(A_1), \dots, \mu_N(A_N)),$$

holds for all nonempty sets $A_1 \subset \Omega_1, \dots, A_N \subset \Omega_N$
if and only if
for almost all $x_1, \dots, x_N$, for all $i = 1, \dots, N, k = 1, \dots n$ and $\rho_i^k > 0$, we have

$$f_0 \circ \phi(x_1, \dots, x_N) \prod_{k=1}^{n} \sum_{i=1}^{N} \rho_i^k \frac{\partial \phi_j}{\partial x_i^k} \geq \Phi\left(f_1(x_1) \prod_{k=1}^{n} \rho_1^k, \dots, f_N(x_N) \prod_{k=1}^{n} \rho_N^k \right).$$

Of course, the sets are not necessarily measurable. This is why the measures have to be understood as inner measures. By the inner measure associated with μ we mean $\mu^*(A) = \sup\{\mu(K), K \subset A, K \text{ compact}\}$ defined for any set A. Borell's proof followed the argument of Hadwiger and Ohman [57] and Dinghas [34]. The papers of Das Gupta [32] and of Prékopa [83] illuminate very much the situation. It is now well understood that we can prove the Prékopa-Leindler inequality (Theorem 3.6) using a parametrisation argument like we have used in the proof of Theorem 2.2. We refer to [13] for an exhaustive presentation. Fradelizi (see [44]) kindly indicated to us that this argument can also be followed for proving Theorem 3.25. Theorem 3.25 is extremely important, not only in the log-concave case but also in the s-concave setting, $s \in \mathbb{R}$. The case $s < 0$ is also known in the literature as the case of convex measures or unimodal functions.

The geometric consequences of these functional inequalities are now classical. Theorem 3.15 is due to Prékopa [82]. Proposition 3.16 appeared first in [33]. Proposition 3.20 and Corollary 3.21 are due to Rogers and Shephard [84] and Theorem 3.23 is due to Ball [6].

There was a big amount of work to develop the functional forms of some classical convex geometric inequalities and we refer the interested reader to [4, 45, 46, 66, 67, 47].

4 Concentration of measure. Dvoretzky's Theorem.

4.1 Isoperimetric problem

The Brunn-Minkowski inequality yields the isoperimetric inequality for the Lebesgue measure on $\mathbb{R}^n$. Indeed, suppose we have a compact set $A \subset \mathbb{R}^n$ and let B be a Euclidean ball of the radius r_A such that $|B| = |A|$. Then from the Brunn-Minkowski inequality we have

$$\begin{aligned} |A_\varepsilon|^{1/n} &= |A + \varepsilon B_2^n|^{1/n} \geq |A|^{1/n} + |\varepsilon B_2^n|^{1/n} \\ &= |B_2^n|^{1/n} r_A + |B_2^n|^{1/n}\varepsilon = |B + \varepsilon B_2^n|^{1/n} = |B_\varepsilon|^{1/n}. \end{aligned}$$

In general, an isoperimetric problem reads as follows.

Isoperimetric problem. Let (Ω, d) be a metric space and let μ be a Borel measure on Ω. Let $\alpha > 0$ and $\varepsilon > 0$. We set

$$A_\varepsilon = \{x \in \Omega,\ d(x, A) \leq \varepsilon\}.$$

What are the sets $A \subset \Omega$ of the measure α such that

$$\mu(A_\varepsilon) = \inf_{\mu(B)=\alpha} \mu(B_\varepsilon).$$

This problem is very difficult in general. It has been solved in a few cases. For example, as we have seen, the case of $\mathbb{R}^n$ equipped with the Lebesgue measure and the Euclidean distance follows from the Brunn-Minkowski inequality. For the spherical and the Gaussian settings the isoperimetry is also known. These two examples will lead us to the notion of the concentration of measure.

We start with the spherical case (S^{n-1}, d, σ_n) where d is the geodesic metric and σ_n is the Haar measure on S^{n-1}.

4.1 Theorem. *For all $0 < \alpha < 1$ and all $\varepsilon > 0$,*

$$\min\{\sigma_n(A_\varepsilon),\ \sigma_n(A) = \alpha\}$$

is attained for a spherical cap $C = \{x \in S^{n-1}, d(x, x_0) \leq r\}$ with $x_0 \in S^{n-1}, r > 0$, such that $\sigma(C) = \alpha$.

A crucial consequence of Theorem 4.1 is the concentration of measure phenomenon on S^{n-1}. Indeed, if $\alpha = \frac{1}{2}$ then the spherical cap of measure $1/2$ is a half sphere. A simple exercise consists in showing that

$$\sigma_n((C(x_0, r))_\varepsilon^c) \leq \sqrt{\frac{\pi}{8}} \exp(-(n-2)\varepsilon^2/2).$$

It is now easy to deduce the following Corollary.

4.2 Corollary. *If A is a Borel set on S^{n-1} such that $\sigma_n(A) \geq 1/2$ then*

$$\sigma_n(A_\varepsilon) \geq 1 - \sqrt{\frac{\pi}{8}} \exp\left(-(n-2)\varepsilon^2/2\right).$$

We can therefore deduce the concentration of Lipschitz functions on the Euclidean sphere. The statement of this result may be considered as the starting point of the concentration of measure phenomenon. It tells that any 1-Lipschitz function on the sphere of high dimension may be viewed as "constant" when looking at its behaviour on sets of overwhelming measure. Of course the statement is interesting in large dimension.

4.3 Corollary. *Let $f : S^{n-1} \to \mathbb{R}$ be 1-Lipschitz with respect to the geodesic distance. If M is a median of f, namely $\sigma_n(\{f \geq M\}) \geq \frac{1}{2}$ and $\sigma_n(\{f \leq M\}) \geq \frac{1}{2}$, then for $\varepsilon > 0$*

$$\sigma_n(\{f \geq M + \varepsilon\}) \leq \sqrt{\frac{\pi}{8}} \exp(-n\varepsilon^2/4), \ \textit{and}\ \sigma_n(\{f \leq M - \varepsilon\}) \leq \sqrt{\frac{\pi}{8}} \exp(-n\varepsilon^2/4),$$

Moreover,

$$\sigma_n(\{|f - M| \geq \varepsilon\}) \leq \sqrt{\frac{\pi}{2}} \exp(-n\varepsilon^2/4).$$

We also know the solution of the isoperimetric problem in the Gaussian setting. Let $\mathbb{R}^n$ be equipped with the Euclidean distance $|\cdot|_2$ and γ_n be the standard Gaussian distribution

$$\mathrm{d}\gamma_n(x) = e^{-|x|_2^2/2} \frac{\mathrm{d}x}{(2\pi)^{n/2}}.$$

Let Φ be the distribution function of γ_1, i.e., we define for any $u \in \mathbb{R}$

$$\Phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-t^2/2}\, \mathrm{d}t.$$

4.4 Theorem. *Let $a \in \mathbb{R}$ and let A be a Borel set in $\mathbb{R}^n$ such that $\gamma_n(A) = \Phi(a)$, then*

$$\gamma_n(A_\varepsilon) \geq \Phi(a + \varepsilon).$$

The theorem tells that half spaces are solutions of the isoperimetric problem that is $\gamma_n(A_\varepsilon) \geq \gamma_n(H_\varepsilon)$, whenever $\gamma_n(H) = \gamma_n(A) = \Phi(a)$, and for some $\theta \in S^{n-1}$, $H = \{x \in \mathbb{R}^n,\ \langle x, \theta \rangle \leq a\}$ is a half space.

As before, having this isoperimetric result at hand, we deduce results concerning the concentration of measure phenomenon in the Gaussian setting. Since for any $r > 0$ we have

$$1 - \Phi(r) \leq \frac{1}{2} e^{-r^2/2}$$

it is easy to deduce the following corollary.

4.5 Corollary. *If $A \subset \mathbb{R}^n$ satisfies $\gamma_n(A) \geq 1/2$ then*

$$\gamma_n(A_r) \geq 1 - \frac{1}{2} e^{-r^2/2}.$$

Moreover, if $F : \mathbb{R}^n \to \mathbb{R}$ is a 1-Lipschitz function with respect to $|\cdot|_2$ and M is a median of F then

$$\gamma_n(\{F \geq M + r\}) \leq \frac{1}{2} e^{-r^2/2}, \quad \gamma_n(\{F \leq M - r\}) \leq \frac{1}{2} e^{-r^2/2}$$

and

$$\gamma_n(\{|F - M| \geq r\}) \leq e^{-r^2/2}.$$

4.2 Concentration inequalities

In many applications we just want concentration inequalities and we do not care much about the constants. This is why we are interested in presenting simpler proofs of these concentration inequalities, which may lead to more general results. We start off by proving the following simple and deep inequality.

4.6 Theorem. *Let $A \subset \mathbb{R}^n$ and let γ_n be the Gaussian measure. Then*

$$\int \exp\left(\frac{d(x,A)^2}{4}\right) \mathrm{d}\gamma_n(x) \leq \frac{1}{\gamma_n(A)}. \tag{4.1}$$

Moreover, if $\gamma_n(A) \geq 1/2$ then

$$\gamma_n(A_\varepsilon) \geq 1 - 2\exp(-\varepsilon^2/4). \tag{4.2}$$

Proof. Let

$$f(x) = \frac{1}{(2\pi)^{n/2}} \exp(d(x,A)^2/4) \exp(-|x|_2^2/2),$$

$$g(y) = \frac{1}{(2\pi)^{n/2}} \mathbf{1}_A(y) \exp(-|y|_2^2/2)$$

and

$$h(z) = \frac{1}{(2\pi)^{n/2}} \exp(-|z|_2^2/2).$$

We show that

$$h\left(\frac{x+y}{2}\right) \geq \sqrt{f(x)}\sqrt{g(y)}.$$

Indeed, it suffices to consider the case when $y \in A$. In this case we have $d(x, A) \leq |x-y|_2$ and therefore

$$\begin{aligned}(2\pi)^n f(x)g(y) &\leq \exp\left(\frac{|x-y|_2^2}{4} - \frac{|x|_2^2}{2} - \frac{|y|_2^2}{2}\right) = \exp\left(-\frac{|x+y|_2^2}{4}\right) \\ &= (2\pi)^n \left(h\left(\frac{x+y}{2}\right)\right)^2.\end{aligned}$$

By the Prékopa-Leindler inequality we obtain

$$1 = \left(\int h\right)^2 \geq \left(\int f\right)\left(\int g\right) = \gamma_n(A) \int \exp\left(\frac{d(x,A)^2}{4}\right) \mathrm{d}\gamma_n(x).$$

The second part of the statement follows from Markov's inequality. Indeed, if $\gamma_n(A) \geq 1/2$ then

$$\int \exp(d(x,A)^2/4)\mathrm{d}\gamma_n(x) \leq 2,$$

hence

$$\gamma_n(d(x,A) \geq \varepsilon) \leq \exp(-\varepsilon^2/4) \int \exp\left(\frac{d(x,A)^2}{4}\right) \mathrm{d}\gamma_n(x) \leq 2\exp(-\varepsilon^2/4).$$

□

As usual, it is now easy to deduce the concentration of measure phenomenon for 1-Lipschitz functions.

4.7 Corollary. *If M is a γ_n median of a 1-Lipschitz function f, then*

$$\gamma_n(\{f \geq M + \varepsilon\}) \leq 2\exp(-\varepsilon^2/4), \ \gamma_n(\{f \leq M - \varepsilon\}) \leq 2\exp(-\varepsilon^2/4),$$

and

$$\gamma_n(\{|f - M| \geq \varepsilon\}) \leq 4\exp(-\varepsilon^2/4).$$

Proof. Let $A = \{f \leq M\}$. Then $\gamma_n(A) \geq 1/2$. Since f is 1-Lipschitz we have $\{f \geq M + \varepsilon\} \subset A_\varepsilon^c$. Therefore,

$$\gamma_n(\{f \geq M + \varepsilon\}) \leq \gamma_n(A_\varepsilon^c) \leq 2\exp(-\varepsilon^2/4).$$

The second inequality is proven identically, taking $A = \{f \leq M\}$. □

Sometimes, it is not so easy to use a concentration inequality with respect to the median of the function. Historically, there is another way to prove Gaussian concentration inequalities in the setting of random vectors in a Banach space. For $a_1, \ldots, a_k$ in a Banach space E and $g_1, \dots, g_k$ i.i.d. standard Gaussian random variables $\mathcal{N}(0,1)$, we define a Gaussian vector

$$X = \sum_{i=1}^k g_i a_i \in E.$$

We define the operator $u : \ell_2^k \to E$ by $u(e_i) = a_i$, where $(e_i)_{i=1}^k$ is the standard orthonormal basis in ℓ_2^k. The *weak variance* of X is $\sigma(X) = \left\| u : \ell_2^k \to E \right\|$. Observe that

$$\sigma(X) = \sup_{|x|_2 \le 1} \|u(x)\| = \sup_{|x|_2 \le 1} \sup_{\substack{\xi \in E^\star \\ \|\xi\|_\star \le 1}} |\xi(u(x))| = \sup_{\substack{\xi \in E^\star \\ \|\xi\|_\star \le 1}} \sup_{|x|_2 \le 1} |\xi(u(x))|.$$

Writing $x = \sum_{i=1}^k x_i e_i$ so that $|x|_2^2 = \sum x_i^2$, we deduce that

$$\sup_{|x|_2 \le 1} |\xi(u(x))| = \sup_{|x|_2 \le 1} \left| \xi \left(\sum_{i=1}^k x_i a_i \right) \right| = \sup_{|x|_2 \le 1} \left| \sum_{i=1}^k x_i \xi(a_i) \right| = \left(\sum_{i=1}^k |\xi(a_i)|^2 \right)^{1/2}$$

and consequently,

$$\sigma(X) = \sup_{\|\xi\|_\star \le 1} \left(\sum_{i=1}^k |\xi(a_i)|^2 \right)^{1/2}.$$

We present now a Gaussian concentration inequality of a Lipschitz function around its mean. The argument is based on the study of a Gaussian process. The important fact is that if X_1, X_2 are two independent copies of a Gaussian vector, then for all $\theta \in \mathbb{R}$ we have the equality in law

$$(X_1 \cos\theta + X_2 \sin\theta, -X_1 \sin\theta + X_2 \cos\theta) \sim (X_1, X_2).$$

4.8 Theorem. *For a Gaussian vector $X = \sum_{i=1}^k g_i a_i$ with values in a Banach space E we have*

$$\mathbb{P}\left(\Big| \|X\| - \mathbb{E}\|X\| \Big| > t \right) \le 2 \exp\left(-\frac{2t^2}{\pi^2 \sigma(X)^2} \right). \tag{4.3}$$

Proof. Let $F : \mathbb{R}^k \to \mathbb{R}$ be given by the formula

$$F(x) = \|u(x)\| = \left\| \sum_{i=1}^k x_i a_i \right\|$$

and let G_1, G_2 be two independent copies of the standard Gaussian vector $(g_1, \dots, g_k)$. We take

$$G(\theta) = G_1 \cos\theta + G_2 \sin\theta.$$

Observe that $G(0) = G_1$, $G(\pi/2) = G_2$ and

$$G'(\theta) = -G_1 \sin\theta + G_2 \cos\theta.$$

Therefore,

$$(G(\theta), G'(\theta)) \sim (G_1, G_2).$$

The function F is Lipschitz and therefore it is absolutely continuous, so we can apply the fundamental theorem of calculus. Alternatively, one can approximate F by C^1 functions. We have

$$F(G_2) - F(G_1) = \int_0^{\pi/2} \frac{\mathrm{d}}{\mathrm{d}\theta} F(G(\theta))\,\mathrm{d}\theta = \int_0^{\pi/2} \langle \nabla F(G(\theta)), G'(\theta)\rangle\,\mathrm{d}\theta.$$

Jensen's inequality for the convex function exp and the normalized Lebesgue measure on $[0, \pi/2]$ yields for every $\lambda > 0$,

$$\begin{aligned}\exp(\lambda(F(G_2) - F(G_1))) &= \exp\left(\frac{2}{\pi}\int_0^{\pi/2} \lambda\frac{\pi}{2}\langle \nabla F(G(\theta)), G'(\theta)\rangle\,\mathrm{d}\theta\right)\\ &\le \frac{2}{\pi}\int_0^{\pi/2} \exp\left(\lambda\frac{\pi}{2}\langle \nabla F(G(\theta)), G'(\theta)\rangle\right)\mathrm{d}\theta.\end{aligned}$$

Taking expectation we deduce

$$\mathbb{E}_{G_1}\mathbb{E}_{G_2}\exp(\lambda(F(G_2) - F(G_1))) \le \frac{2}{\pi}\int_0^{\pi/2} \mathbb{E}_{G_1}\mathbb{E}_{G_2}\exp\left(\lambda\frac{\pi}{2}\langle \nabla F(G(\theta)), G'(\theta)\rangle\right)\mathrm{d}\theta$$

But the function

$$\theta \mapsto \mathbb{E}_{G_1}\mathbb{E}_{G_2}\exp\left(\lambda\frac{\pi}{2}\langle \nabla F(G(\theta)), G'(\theta)\rangle\right)$$

is a constant function, since $(G(\theta), G'(\theta)) \sim (G_1, G_2)$. Therefore,

$$\begin{aligned}\mathbb{E}_{G_1}\mathbb{E}_{G_2}\exp(\lambda(F(G_2) - F(G_1))) &\le \mathbb{E}_{G_1}\mathbb{E}_{G_2}\exp\left(\lambda\frac{\pi}{2}\langle \nabla F(G_1), G_2\rangle\right)\\ &= \mathbb{E}_{G_1}\exp\left(\lambda^2\pi^2|\nabla F(G_1)|_2^2/8\right),\end{aligned}$$

where we have computed the expectation over G_2.

Recall that $\|u\|_{\ell_2^k \to E} = \sigma(X)$. We obtain

$$|F(x) - F(y)| = |\,\|u(x)\| - \|u(y)\|\,| \le \|u(x-y)\| \le \sigma(X)|x-y|_2,$$

therefore $|\nabla F(\cdot)|_2 \le \sigma(X)$. We then arrive at

$$\mathbb{E}_{G_1}\mathbb{E}_{G_2}\exp(\lambda(F(G_2) - F(G_1))) \le \exp\left(\frac{\lambda^2\pi^2\sigma(X)^2}{8}\right).$$

Using Jensen's inequality to the convex function $\exp(-(\cdot))$ and the expectation over G_1 we have

$$\begin{aligned}\mathbb{E}_{G_1}\mathbb{E}_{G_2}\exp(\lambda(F(G_2)-F(G_1))) &\geq \mathbb{E}_{G_2}\exp\left(\lambda(F(G_2)-\mathbb{E}_{G_1}F(G_1))\right)\\ &= \mathbb{E}\exp\left(\lambda(\|X\|-\mathbb{E}\,\|X\|)\right).\end{aligned}$$

Therefore,

$$\mathbb{E}\exp\left(\lambda(\|X\|-\mathbb{E}\,\|X\|)\right) \leq \exp\left(\frac{\lambda^2\pi^2\sigma(X)^2}{8}\right).$$

Using Markov's inequality we get

$$\mathbb{P}\left(\|X\|-\mathbb{E}\,\|X\|>t\right) \leq \inf_{\lambda>0}\exp\left(-\lambda t+\frac{\lambda^2\pi^2\sigma(X)^2}{8}\right) = \exp\left(-\frac{2t^2}{\pi^2\sigma(X)^2}\right).$$

□

We have seen that the Lipschitz constant of $F:\mathbb{R}^k\to\mathbb{R}$ defined by $F(x)=\left\|\sum_{i=1}^k x_i a_i\right\|$ is $\sigma(X)$ hence Corollary 4.7 gives a concentration inequality of F around its median (which is not very different than a concentration inequality around its mean).

4.9 Remark. Of course the same argument yields that if $F:\mathbb{R}^k\to\mathbb{R}$ is L-Lipschitz with respect to the Euclidean norm on $\mathbb{R}^k$ then for every $t>0$,

$$\mathbb{P}\left((|F(G)-\mathbb{E}F(G)|>t)\right) \leq 2\exp\left(-\frac{2t^2}{\pi^2L^2}\right),$$

where $G\sim\mathcal{N}(0,\mathrm{Id})$ is a standard Gaussian vector in $\mathbb{R}^k$.

We now give an improvement of this result, based on the same method of proof.

4.10 Theorem. *Let $G_\omega:\ell_2^k\to\mathbb{R}^n$ be a random Gaussian operator given by an $n\times k$ matrix with the independent standard Gaussian entries. Let $a,b\in S^{k-1}$ and let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ such that $\|\cdot\|\leq|\cdot|_2$. Then*

$$\mathbb{P}\left(\|G_\omega(a)\|-\|G_\omega(b)\|\geq t\right) \leq \exp\left(-\frac{2t^2}{\pi^2|a-b|_2^2}\right). \tag{4.4}$$

4.11 Corollary. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ such that for any $x\in\mathbb{R}^n$, $\|x\|\leq|x|_2$. For any set $T\subset S^{k-1}$, we have*

$$\mathbb{E}\sup_{a\in T}\left|\|G_\omega(a)\|-\mathbb{E}\|G_{(n)}\|\right| \leq C\,\mathbb{E}\sup_{a\in T}|\langle G_{(k)},a\rangle|,$$

where $G_{(n)}$ and $G_{(k)}$ are standard Gaussian vectors in $\mathbb{R}^n$ and $\mathbb{R}^k$, and where C is a universal constant. In particular,

$$\mathbb{E}\|G_{(n)}\|-C\,\mathbb{E}|G_{(k)}|_2 \leq \mathbb{E}\inf_{a\in S^{k-1}}\|G_\omega(a)\| \leq \mathbb{E}\sup_{a\in S^{k-1}}\|G_\omega(a)\| \leq \mathbb{E}\|G_{(n)}\|+C\,\mathbb{E}|G_{(k)}|_2.$$

Proof. If $G_\omega = (G_1, \ldots, G_k) = (g_{ij})$ where $1 \le i \le n$, $1 \le j \le k$ then

$$G_\omega(a) = \sum_{j=1}^{k} a_j G_j = \sum_{i=1}^{n} \left(\sum_{j=1}^{k} a_j g_{ij} \right) e_i,$$

for $a \in S^{n-1}$. Therefore $G_\omega(a)$ has the same distribution as the standard Gaussian vector $G_{(n)} = (g_1, \ldots, g_n)$ on $\mathbb{R}^n$. Indeed, we only have to check the covariance matrix,

$$\mathbb{E} \left(\sum_{j=1}^{k} a_j g_{kj} \right) \left(\sum_{j=1}^{k} a_j g_{lj} \right) = \sum_{j=1}^{k} \sum_{j'=1}^{k} a_j a_{j'} \mathbb{E} g_{kj} g_{lj'} = \sum_{j=1}^{k} \sum_{j'=1}^{k} a_j a_{j'} \delta_{k,l} \delta_{j,j'}$$
$$= \delta_{k,l} \sum_{j=1}^{k} a_j^2 = \delta_{k,l},$$

where we use Kronecker's delta $\delta_{k,l} = 1$ if and only if $k = l$. Hence $\mathbb{E} \|G_\omega(a)\| = \mathbb{E} \left\|G_{(n)}\right\|$. Theorem 4.10 tells that the random process

$$Y : a \mapsto \|G_\omega(a)\| - \mathbb{E} \|G_\omega(a)\| = \|G_\omega(a)\| - \mathbb{E} \left\|G_{(n)}\right\|$$

is a subgaussian process, namely

$$\mathbb{P}\left(Y(a) - Y(b) > t\right) \le \exp\left(-\frac{2t^2}{\pi^2 |a-b|_2^2} \right).$$

One can therefore apply the majorizing measure theorem [90] to deduce that for any set $T \subset S^{k-1}$, we have

$$\mathbb{E} \sup_{a \in T} |Y(a)| \le C\, \mathbb{E} \sup_{a \in T} | \left\langle G_{(k)}, a \right\rangle |,$$

where C is a universal constant. The particular case is obtained by taking $T = S^{k-1}$. It can be checked that

$$\mathbb{E}|G_{(k)}|_2 = \mathbb{E} \left(\sum_{i=1}^{k} g_i^2 \right)^{1/2} \sim \sqrt{k} \ \text{ as } k \to \infty.$$

□

Proof of Theorem 4.10. We follow the same idea as in the proof of Theorem 4.8. For $a, b \in S^{k-1}$ we set $X_a = G_\omega(a)$ and $X_b = G_\omega(b)$. We can find a vector a' such that $a \perp a'$ and $b = a\cos\theta_0 + a'\sin\theta_0$ with $\theta_0 \in [0, \pi]$. Let $X_{a'} = G_\omega(a')$. We take

$$X(\theta) = X_a \cos\theta + X_{a'} \sin\theta.$$

Since G_ω is a linear operator, we have

$$X(\theta_0) = G_\omega(b) = X_b.$$

We take $F(x) = \|x\|$. Then using Jensen's inequality

$$\begin{aligned}\mathbb{E}\exp\left(\lambda(\|G_\omega(b)\| - \|G_\omega(a)\|)\right) &= \mathbb{E}\exp\left(\lambda\left(F(X_a\cos\theta_0 + X_{a'}\sin\theta_0) - F(X_a)\right)\right) \\ &\le \frac{1}{\theta_0}\int_0^{\theta_0}\mathbb{E}\exp\left(\lambda\theta_0\left\langle\nabla F(X(\theta)), X'(\theta)\right\rangle\right)\mathrm{d}\theta \\ &= \mathbb{E}\exp\left(\lambda\theta_0\left\langle\nabla F(X_a), X_{a'}\right\rangle\right) \\ &\le \exp(\lambda^2\theta_0^2/2),\end{aligned}$$

for

$$|F(x) - F(y)| = |\|x\| - \|y\|| \le \|x - y\| \le |x - y|_2.$$

Now it suffices to observe that

$$|a - b|_2^2 = 2(1 - \cos\theta_0) = 4\sin^2(\theta_0/2) \ge 4(2/\pi)^2(\theta_0^2/4),$$

and to conclude with Markov's inequality as it is done in the proof of Theorem 4.8. □

4.3 Dvoretzky's Theorem

We denote by $\mathcal{G}_{n,k}$ the set of k-dimensional subspaces of $\mathbb{R}^n$ equipped with its Haar measure, that is the unique probability measure invariant under the action of the orthogonal group on $\mathbb{R}^n$. Dvoretzky's theorem tells about the random Euclidean sections of a symmetric convex body in $\mathbb{R}^n$.

4.12 Theorem. *Let K be a symmetric convex body in $\mathbb{R}^n$ such that $B_2^n \subset bK$. Let M be a median of $\|\cdot\|$ with respect to σ_n on S^{n-1}, where $\|\cdot\| = \|\cdot\|_K$. Then for every $\varepsilon \in (0,1)$, if*

$$k = \left\lfloor \frac{cnM^2\varepsilon^2}{b^2\ln(4/\varepsilon)} \right\rfloor$$

then the set of subspaces $E \in \mathcal{G}_{n,k}$ such that

$$(1-\varepsilon)\,M\,(K\cap E)\ \subset\ B_2^n \cap E\ \subset\ (1+\varepsilon)\,M\,(K\cap E)$$

has a measure greater than

$$1 - 2\exp\left(-k\log\left(\frac{c}{\varepsilon}\right)\right).$$

Here $c > 0$ is an absolute constant.

We will prove the Gaussian version of this theorem.

4.13 Theorem. *Let K be a symmetric convex body in $\mathbb{R}^n$ such that $B_2^n \subset bK$. Let $\|\cdot\|$ be the norm associated with K and let G be a standard Gaussian vector in $\mathbb{R}^n$. Then for every $\varepsilon > 0$, if*

$$k = \left\lfloor \frac{(\mathbb{E}\,\|G\|)^2\varepsilon^2}{b^2\pi^2\ln(21/\varepsilon)} \right\rfloor$$

then the set of subspaces $E \in \mathcal{G}_{n,k}$ such that

$$(1-\varepsilon)\frac{\mathbb{E}\,\|G\|}{\mathbb{E}|G|_2}\,(K\cap E) \ \subset\ B_2^n \cap E \ \subset\ (1+\varepsilon)\frac{\mathbb{E}\,\|G\|}{\mathbb{E}|G|_2}\,(K\cap E) \tag{4.5}$$

has a measure greater than

$$1 - 4\exp\left(-k\log\left(\frac{21}{\varepsilon}\right)\right).$$

4.14 Remark. Let θ be a random vector uniformly distributed on the unit sphere S^{n-1}. Then $G \sim |G|_2\theta$, hence

$$\frac{\mathbb{E}\,\|G\|}{\mathbb{E}|G|_2} = \mathbb{E}\,\|\theta\|\,.$$

Thus, $\mathbb{E}\,\|G\| \approx \sqrt{n}\mathbb{E}\,\|\theta\|$, so up to the fact that M is replaced with $\mathbb{E}\,\|\theta\| = \int_{S^{n-1}}\|\theta\|\,\mathrm{d}\sigma_n(\theta)$, both theorems are identical.

The idea of the proof is standard now. We consider the random Gaussian operator $G_\omega : \ell_2^k \to (\mathbb{R}^n, \|\cdot\|)$ and we

a) do an individual estimate on deviations of $\|G_\omega(\alpha)\|$ from its mean,

b) apply a discretization argument (construct a net),

c) deduce a general estimate from a net estimate.

4.15 Remark. A procedure to generate the Haar measure $\nu_{n,k}$ on $\mathcal{G}_{n,k}$ is the following. Let γ_n be the standard $\mathcal{N}(0,\mathrm{Id})$ Gaussian measure on $\mathbb{R}^n$. Take the push-forward of the product of $\gamma_n \times \ldots \times \gamma_n$ on $\mathbb{R}^n \oplus \ldots \oplus \mathbb{R}^n$ under the map $\mathrm{span}\{x_1,\ldots,x_k\}$. The result is invariant under the action of the orthogonal group and it has to be the Haar measure on $\mathcal{G}_{n,k}$ because of the uniqueness. If we denote by $\mathcal{A}$ the subspaces of $\mathcal{G}_{n,k}$ such that (4.5) holds true then $\nu_{n,k}(E \in \mathcal{A}) = \mathbb{P}(\mathrm{Im}\,G_\omega \in \mathcal{A})$

4.16 Lemma. *For every $\delta \in (0,1)$ there exists a δ-net of S^{k-1} with respect to $|\cdot|_2$ of cardinality less than $(3/\delta)^k$.*

Proof. Let $\theta_1,\ldots,\theta_M$ be a maximal number of points of S^{k-1} such that for all $i \neq j$, $|\theta_i - \theta_j| > \delta$. Then, for any $\theta \in S^{k-1}$, there exists $i \in \{1,\ldots,M\}$ such that $|\theta - \theta_i| \leq \delta$, otherwise, the set would not have been maximal. Hence $\{\theta_1,\ldots,\theta_M\}$ is a δ-net of S^{k-1}.

It remains to estimate M. The Euclidean balls centred at θ_i of radius $\delta/2$ are disjoint. They are all contained in the Euclidean ball centred at the origin and of radius $1+\frac{\delta}{2}$. We get

$$\begin{aligned}\mathrm{vol}\left(\bigcup_{i=1}^{M} B\left(\theta_i,\frac{\delta}{2}\right)\right) &= \sum_{i=1}^{M}\mathrm{vol}\left(B\left(\theta_i,\frac{\delta}{2}\right)\right) = M\left(\frac{\delta}{2}\right)^k \mathrm{vol}(B_2^n)\\ &\le \left(1+\frac{\delta}{2}\right)^k \mathrm{vol}(B_2^n)\end{aligned}$$

which proves that $M \le (1+2/\delta)^k \le (3/\delta)^k$. □

4.17 Lemma. *Let $\mathcal{N}$ be a δ-net of S^{k-1} with respect to $|\cdot|_2$ and let $T:\ell_2^k \to (\mathbb{R}^n,\|\cdot\|)$ be an operator such that $\lambda_2 \le \|T\alpha\| \le \lambda_1$ for all $\alpha\in\mathcal{N}$. Then for all $x\in S^{k-1}$ we have*

$$\lambda_2 - \frac{\delta\lambda_1}{1-\delta} \le \|Tx\| \le \frac{\lambda_1}{1-\delta}.$$

Proof. Let $x_0\in S^{k-1}$ be such that $\|Tx_0\| = \max_{x\in S^{k-1}}\|Tx\|$. There exists an element α_0 of the δ-net $\mathcal{N}$ such that $|\alpha_0 - x_0|_2 \le \delta$. We have

$$\|Tx_0\| \le \|T\alpha_0\| + \|T(x_0-\alpha_0)\| \le \lambda_1 + |\alpha_0-x_0|_2\left\|T\left(\frac{\alpha_0-x_0}{|\alpha_0-x_0|_2}\right)\right\| \le \lambda_1 + \delta\|Tx_0\|,$$

hence $\|Tx_0\|\le \lambda_1/(1-\delta)$. Now let $x\in S^{k-1}$ and take $\alpha\in\mathcal{N}$ such that $|\alpha-x|_2\le\delta$. Then

$$\frac{\lambda_1}{1-\delta} \ge \|Tx_0\| \ge \|Tx\| \ge \|T\alpha\| - \|T(x-\alpha)\| \ge \lambda_2 - \frac{\delta\lambda_1}{1-\delta}.$$

□

Proof of Theorem 4.13. If $G=\sum_{i=1}^n g_i e_i$ then with $E=(\mathbb{R}^n,\|\cdot\|)$ we deduce from $B_2^n\subset bK$ that

$$\sigma(G) = \|id:\ell_2^n\to(\mathbb{R}^n,\|\cdot\|)\| \le b.$$

Set $a\in S^{k-1}$. Since $G_\omega(a)\sim G$ then by Theorem 4.8 we have

$$\mathbb{P}\left(\Big|\,\|G_\omega(a)\| - \mathbb{E}\|G_\omega(a)\|\,\Big| > t\right) \le 2\exp\left(-\frac{ct^2}{b^2}\right),$$

where we can set $c=2/\pi^2$. Therefore,

$$\mathbb{P}\left(\Big|\,\|G_\omega(a)\| - \mathbb{E}\|G\|\,\Big| > \varepsilon\mathbb{E}\|G\|\right) \le 2\exp\left(-\frac{c\varepsilon^2(\mathbb{E}\|G\|)^2}{b^2}\right).$$

Let $\mathcal{N}$ be an ε-net in the unit sphere of cardinality $(3/\varepsilon)^k$. Then the union bound gives

$$\begin{aligned}\mathbb{P}\left(\exists \alpha \in \mathcal{N}\,;\, \Big| \|G_\omega(a)\| - \mathbb{E}\,\|G\| \Big| > \varepsilon \mathbb{E}\,\|G\|\right) &\leq 2|\mathcal{N}| \exp\left(-\frac{c\varepsilon^2(\mathbb{E}\,\|G\|)^2}{b^2}\right)\\ &\leq 2\exp\left(k \ln\left(\frac{3}{\varepsilon}\right) - \frac{c\varepsilon^2(\mathbb{E}\,\|G\|)^2}{b^2}\right)\end{aligned}$$

Then if

$$k \ln\left(\frac{3}{\varepsilon}\right) \leq \frac{1}{2}\cdot \frac{c\varepsilon^2(\mathbb{E}\,\|G\|)^2}{b^2}, \tag{4.6}$$

we have

$$\forall \alpha \in \mathcal{N}, \qquad (1-\varepsilon)\mathbb{E}\,\|G\| \leq \|G_\omega(\alpha)\| \leq (1+\varepsilon)\mathbb{E}\,\|G\|$$

with probability greater than

$$1 - 2\exp\left(-k\ln\left(\frac{3}{\varepsilon}\right)\right).$$

Since $\left(1-\varepsilon-\frac{\varepsilon(1+\varepsilon)}{1-\varepsilon}\right) = \frac{1-3\varepsilon}{1-\varepsilon}$, we deduce from Lemma 4.17 that

$$\forall x \in S^{k-1}, \qquad \frac{1-3\varepsilon}{1-\varepsilon}\,\mathbb{E}\,\|G\| \leq \|G_\omega(x)\| \leq \frac{1+\varepsilon}{1-\varepsilon}\,\mathbb{E}\,\|G\|$$

If k satisfies (4.6) then, thanks to $\|\cdot\| \leq b\,|\cdot|$, we observe that

$$k\ln\left(\frac{3}{\varepsilon}\right) \leq \frac{1}{2}c\varepsilon^2(\mathbb{E}|G|_2)^2$$

and therefore we can get the same conclusion with $\|\cdot\|$ replaced by $|\cdot|_2$,

$$\forall x \in S^{k-1}, \qquad \frac{1-3\varepsilon}{1-\varepsilon}\,\mathbb{E}|G|_2 \leq |G_\omega(x)|_2 \leq \frac{1+\varepsilon}{1-\varepsilon}\,\mathbb{E}|G|_2.$$

Taking the intersection of the two events we infer that with probability greater than

$$1 - 4\exp\left(-k\ln\left(\frac{3}{\varepsilon}\right)\right)$$

both conclusions hold true for the operator G_ω. Using these inequalities and homogeneity of the norm we have

$$\forall x \in \mathbb{R}^k \qquad \frac{1-3\varepsilon}{1+\varepsilon}\cdot\frac{\mathbb{E}\,\|G\|}{\mathbb{E}|G|_2} \leq \frac{\|G_\omega(x)\|}{|G_\omega(x)|_2} \leq \frac{1+\varepsilon}{1-3\varepsilon}\cdot\frac{\mathbb{E}\,\|G\|}{\mathbb{E}|G|_2}$$

with high probability. We set $E = \mathrm{Im}G_\omega$. Therefore, if k satisfies (4.6), that is,

$$k \leq \frac{c(\mathbb{E}\,\|G\|)^2\varepsilon^2}{2b^2\ln(3/\varepsilon)},$$

then we have

$$\forall y \in E \qquad \frac{1-3\varepsilon}{1+\varepsilon} \cdot \frac{\mathbb{E}\,\|G\|}{\mathbb{E}|G|_2} \le \frac{\|y\|}{|y|_2} \le \frac{1+\varepsilon}{1-3\varepsilon} \cdot \frac{\mathbb{E}\,\|G\|}{\mathbb{E}|G|_2}.$$

Moreover, it is clear that $\dim E = k$ and changing ε to $\varepsilon/7$, we achieve our goal. The result follows from Remark 4.15. □

We will need later a dual version of this theorem. We write it for a simple choice of ε.

4.18 Theorem. *Let K be a symmetric convex body in $\mathbb{R}^n$ such that its support function h_K satisfies $h_K(\cdot) \le b\,|\cdot|_2$. If*

$$k \le \frac{c(\mathbb{E}h_K(G))^2}{b^2}$$

then the set of subspaces $E \in \mathcal{G}_{n,k}$ such that

$$\frac{1}{2}\frac{\mathbb{E}h_K(G)}{\mathbb{E}|G|_2} P_E B_2^n \;\subset\; P_E K \;\subset\; \frac{3}{2}\frac{\mathbb{E}h_K(G)}{\mathbb{E}|G|_2} P_E B_2^n$$

has probability greater than

$$1 - 4\exp(-ck),$$

where c is a universal constant.

Proof. Note that $\|\cdot\|_{K^\circ} = h_K$, where h_K is the support function of a convex body K. The hypothesis $\|\cdot\|_{K^\circ} \le b\,|\cdot|_2$ is equivalent to $B_2^n \subset bK^\circ$. Applying Theorem 4.13 to K°, we get that for $\varepsilon = 1/2$, if

$$k \le \frac{c(\mathbb{E}h_K(G))^2}{b^2},$$

then there exists a set of subspaces $E \in \mathcal{G}_{n,k}$ of measure greater than $1 - 4e^{-ck}$ such that

$$\frac{\mathbb{E}h_K(G)}{2\mathbb{E}|G|_2}(K^\circ \cap E) \;\subset\; B_2^n \cap E \;\subset\; \frac{3\mathbb{E}h_K(G)}{2\mathbb{E}|G|_2}(K^\circ \cap E).$$

We can now dualize these inclusions using $(B_2^n \cap E)^\circ = P_E B_2^n$, $(K^\circ \cap E)^\circ = P_E K$ to obtain

$$\frac{\mathbb{E}h_K(G)}{2\mathbb{E}|G|_2} P_E B_2^n \;\subset\; P_E K \;\subset\; \frac{3\mathbb{E}h_K(G)}{2\mathbb{E}|G|_2} P_E B_2^n.$$

□

To conclude this part, we state and prove the classical Dvoretzky's Theorem.

4.19 Theorem. *Let $(\mathbb{R}^n, \|\cdot\|)$ be a normed space. For every $\varepsilon \in (0,1)$, there exists a subspace $E \subset (\mathbb{R}^n, \|\cdot\|)$ of dimension $k \ge c(\varepsilon)\log n$ such that $d(E, \ell_2^k) \le 1+\varepsilon$.*

Consequently, ℓ_2 is finitely representable in any infinite dimensional Banach space X.

Proof. Although in the notes the concept of Banach Mazur distance between two Banach spaces has not been explained, we refer for a basic presentation to classical books on Banach spaces, e.g. [80, 92]. The statement of the theorem means that given $\mathbb{R}^n$ equipped with a norm $\|\cdot\|$ and its unit ball K then one can find a linear transformation $T \in \mathrm{GL}_n$ such that $T(K)$ admits a section with a subspace E of dimension k satisfying

$$r\, B_2^n \cap E \subset T(K) \cap E \subset R\, B_2^n \cap E \quad \text{with } \frac{R}{r} \leq 1 + \varepsilon.$$

Of course, $B_2^n \cap E$ is nothing else but a Euclidean ball in dimension k that you may identify as B_2^k. We can now start the proof.

Let $T \in \mathrm{GL}_n$ be such that B_2^n is the ellipsoid of maximal volume contained in $T(K)$. This map exists and is uniquely characterized by Theorem 2.10. Of course, $B_2^n \subset T(K)$ and by Theorem 2.11 we get

$$\mathbb{E}\|G\|_{T(K)} \geq \mathbb{E}|G|_\infty = \mathbb{E} \max_{1 \leq i \leq n} |g_i| \geq c\sqrt{\log n},$$

where the last inequality follows from a simple estimate of the distribution of the maximum of n independent Gaussian standard $\mathcal{N}(0,1)$ random variables. By Theorem 4.13 with $b = 1$ we conclude that for every $\varepsilon \in (0,1)$ there exists a subspace E of dimension greater than $c\,(\varepsilon^2/\log(1/\varepsilon))\log n$ such that

$$(1-\varepsilon)\frac{\mathbb{E}\|G\|_{T(K)}}{\mathbb{E}|G|_2}\,(T(K)\cap E) \;\subset\; B_2^n \cap E \;\subset\; (1+\varepsilon)\frac{\mathbb{E}\|G\|_{T(K)}}{\mathbb{E}|G|_2}\,(T(K)\cap E)$$

which is the desired conclusion up to a change of ε to $\varepsilon/3$.

By definition, ℓ_2 is finitely representable in an infinite dimensional Banach space X if and only if for any $k \in \mathbb{N}$ and any $\varepsilon \in (0,1)$, there exists a k dimensional subspace $E \subset X$ such that $d(E, \ell_2^k) \leq 1 + \varepsilon$. This follows immediately since $\log n$ goes to infinity as n goes to infinity. □

4.4 Comparison of moments of a norm of a Gaussian vector

From the Gaussian concentration inequalities we can deduce the following theorem.

4.20 Theorem. *There is a constant c such that for any norm $\|\cdot\|$ on $\mathbb{R}^n$ whose unit ball is denoted by K, the following holds true. Assume that $\|\cdot\| \leq b\,|\cdot|_2$. Then*

$$\left(\mathbb{E}\Big|\,\|G\| - \mathbb{E}\,\|G\|\,\Big|^p\right)^{1/p} \leq c\,b\,\sqrt{p}, \quad \text{for } p \geq 1, \tag{4.7}$$

where G is a standard $\mathcal{N}(0,\mathrm{Id})$ Gaussian vector in $\mathbb{R}^n$. Set

$$k^\star(K) = \left(\frac{\mathbb{E}\,\|G\|}{b}\right)^2.$$

If $1 \le p \le k^\star(K)$ *then*

$$1 \le \frac{(\mathbb{E}\,\|G\|^p)^{1/p}}{\mathbb{E}\,\|G\|} \le c. \tag{4.8}$$

If $p > k^\star(K)$ *then*

$$(\mathbb{E}\,\|G\|^p)^{1/p} \le c\,b\,\sqrt{p}. \tag{4.9}$$

In addition, if b *is the smallest constant such that* $\|\cdot\|_K \le b\,|\cdot|_2$*, then, for all* $1 \le p < \infty$ *we have*

$$c\,b\,\sqrt{p} \le (\mathbb{E}\,\|G\|^p)^{1/p}\,. \tag{4.10}$$

Proof. From Theorem 4.8 we deduce

$$\begin{aligned}\mathbb{E}\Big|\,\|G\| - \mathbb{E}\,\|G\|\,\Big|^p &= p\int_0^\infty t^{p-1}\mathbb{P}\left(|\,\|G\| - \mathbb{E}\,\|G\|\,| > t\right)\,\mathrm{d}t \le 2p\int_0^\infty t^{p-1}\exp(-ct^2/b^2)\,\mathrm{d}t\\ &= \frac{p\,b^p}{c^{p/2}}\int_0^\infty u^{\frac{p}{2}-1}\exp(-u)\,\mathrm{d}u = \frac{p\,b^p}{c^{p/2}}\Gamma(p/2).\end{aligned}$$

Therefore to obtain the first inequality it suffices to use Stirling's formula. It follows from the triangle inequality that

$$\Big|(\mathbb{E}\,\|G\|^p)^{1/p} - \mathbb{E}\,\|G\|\,\Big| \le \left(\mathbb{E}\Big|\,\|G\| - \mathbb{E}\,\|G\|\,\Big|^p\right)^{1/p} \le c\,b\,\sqrt{p},$$

therefore if $p \le k^\star(K)$ we have

$$(\mathbb{E}\,\|G\|^p)^{1/p} \le \mathbb{E}\,\|G\| + cb\sqrt{p} \le (1+c)\mathbb{E}\,\|G\|\,.$$

If $p > k^\star(K)$ then

$$(\mathbb{E}\,\|G\|^p)^{1/p} \le \mathbb{E}\,\|G\| + cb\sqrt{p} \le (1+c)b\sqrt{p}.$$

Moreover, for all $1 \le p < \infty$ we have

$$(\mathbb{E}\,\|G\|^p)^{1/p} = (\mathbb{E}\sup_{\|\phi\|_{K^\circ}=1}|\,\langle\phi, G\rangle\,|^p)^{1/p} \ge \sup_{\|\phi\|_{K^\circ}=1}(\mathbb{E}|\,\langle\phi, G\rangle\,|^p)^{1/p}.$$

For any $\phi \in \mathbb{R}^n$, $\langle\phi, G\rangle \sim \mathcal{N}(0, |\phi|_2)$, therefore

$$(\mathbb{E}|\,\langle\phi, G\rangle\,|^p)^{1/p} \ge c\,|\phi|_2\sqrt{p}.$$

If b is the smallest constant such that $\|\cdot\|_K \le b\,|\cdot|_2$ then $|\cdot|_2 \le b\,\|\cdot\|_{K^\circ}$ and there exists $\phi \in \mathbb{R}^n$ such that $|\phi|_2 = b\,\|\phi\|_{K^\circ}$. Therefore, $(\mathbb{E}\,\|G\|^p)^{1/p} \ge c\,b\,\sqrt{p}$. □

4.5 Notes and comments

The results presented in this chapter are at the heart of the study of high dimensional concentration phenomena. They can be considered as the basics of the theory. Theorem 4.1 is due to Lévy [68]. Theorem 4.12 is taken from [75]. Moreover the isoperimetric problem on the sphere is delicate and a proof based on the Steiner symmetrisation can be found in [37]. The paper [37] is a masterpiece on this subject. Vitali Milman had a great influence in the discovery of the power of the concentration measure phenomenon on the Euclidean sphere. His proof of the quantified version of Dvoretzky's Theorem 4.19 is the starting point of a main branch of the local theory of Banach spaces. We emphasize that the original paper of Dvoretzky [35] contains also a quantified finite dimensional version.

In the Gaussian setting, the isoperimetric problem was solved by Sudakov and Tsirelson [89] as well as independently by Borell [23], see Theorem 4.4. It may be deduced from the spherical case by using the Poincaré lemma. There is also a proof by Ehrhard, [36] which uses the so-called Ehrhard symmetrisations.

As we said, all these proofs are delicate and this is why it was attractive to study the concentration of measure phenomenon by itself. We have presented simple proofs in the Gaussian setting. The proof of Theorem 4.8 is due to Maurey and Pisier. We followed the presentation from [80]. Theorem 4.10 is due to Schechtman [85] but we followed a proof indicated to us by Pisier. Up to the constant C, its Corollary 4.11 is know as consequences of Gordon's min-max inequalities [50]. We have showed a proof that uses the Majorizing Measure Theorem of Talagrand [90]. Never mind the original papers of Gordon [50, 51], a detailed proof of the Gordon's min-max inequalities can be found in [69]. Theorem 4.6 is due to Talagrand [91] and we have followed the argument of Maurey [73] using the so-called Property (τ). It can be extended to the setting of uniformly smooth Banach spaces [87, 3] recovering a concentration of measure phenomenon on uniformly convex spaces due to Gromov and Milman [53]. Theorem 4.20 is due to Litvak, Milman and Schechtman [71].

Several books about the concentration of measure phenomenon and its applications have been written. We refer to [65, 64, 25, 30] for further readings about various other results.

5 Reverse Hölder inequalities and volumes of sections of convex bodies

5.1 Berwald's inequality and its extensions

We start by formulating a reverse Hölder inequality due to Berwald [15].

5.1 Theorem. *Let ϕ be a nonnegative concave function supported on a convex body K in $\mathbb{R}^n$. Then for any $0 < p \le q$ we have*

$$\left(\binom{n+p}{n}\frac{1}{|K|}\int_K \phi(x)^p \,\mathrm{d}x\right)^{\frac{1}{p}} \ge \left(\binom{n+q}{n}\frac{1}{|K|}\int_K \phi(x)^q \,\mathrm{d}x\right)^{\frac{1}{q}}.$$

Note that

$$\binom{n+p}{n} = \frac{(n+p)(n+p-1)\dots(p+1)}{n!} = \frac{1}{p\int_0^1 (1-u)^n u^{p-1}\,\mathrm{d}u}. \tag{5.1}$$

Observe also that for any r such that the integrals are finite,

$$\frac{1}{|K|}\int_K \phi(x)^{r+1}\,\mathrm{d}x = (r+1)\int_0^{+\infty} t^r \mu(\{\phi \ge t\})\,\mathrm{d}t, \tag{5.2}$$

where μ is the measure uniformly distributed on K, $\mu(A) = |K \cap A|/|K|$. Since ϕ is concave we have

$$\{\phi \ge (1-\lambda)u + \lambda v\} \supset (1-\lambda)\{\phi \ge u\} + \lambda\{\phi \ge v\}.$$

Let us define $f(t) = \mu(\{\phi \ge t\})$. Since K is a convex body, the measure μ satisfies the Brunn-Minkowski inequality, and by Theorem 3.2, we have

$$f^{1/n}((1-\lambda)u + \lambda v) \ge (1-\lambda)f^{1/n}(u) + \lambda f^{1/n}(v) \tag{5.3}$$

whenever $f(u)f(v) > 0$. We state a generalisation of Berwald's inequality.

5.2 Lemma. *Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a decreasing function. Let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be such that $\Phi(0) = 0$ and the function $x \mapsto \Phi(x)/x$ is increasing. Then the function*

$$G(p) = \left(\frac{\int_0^{+\infty} h(\Phi(x))x^p\,\mathrm{d}x}{\int_0^{+\infty} h(x)x^p\,\mathrm{d}x}\right)^{\frac{1}{p+1}}$$

is decreasing on $(-1, \infty)$.

It is a generalisation of Berwald's inequality. First we show how it implies Theorem 5.1 and then we prove the lemma.

Proof of Theorem 5.1. Let $h(u) = (1-u)^n \mathbf{1}_{[0,1]}(u)$. Take

$$\Phi(x) = 1 - \mu(\{\phi \geq x\})^{1/n},$$

where μ is uniformly distributed on K. By inequality (5.3), we know that Φ is convex. Obviously, $\Phi(0) = 0$. Thus, $\frac{\Phi(x)}{x} = \frac{\Phi(x)-\Phi(0)}{x-0}$ is increasing. Hence, from Lemma 5.2, the function

$$G(p) = \left(\frac{\int_0^{+\infty} \mu(\{\phi \geq x\}) x^p \, \mathrm{d}x}{B(n+1, p+1)} \right)^{\frac{1}{p+1}}$$

is decreasing on $(-1, +\infty)$. Here and throughout we shall use the Beta function $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} \mathrm{d}t$. It follows from (5.1) and (5.2) that

$$\binom{n+p+1}{n} \frac{1}{|K|} \int_K \phi(x)^{p+1} \, \mathrm{d}x = (p+1) \binom{n+p+1}{n} \int_0^{+\infty} t^p \mu(\{\phi \geq t\}) \, \mathrm{d}t = G(p)^{p+1},$$

and Berwald's inequality is proved. □

Proof of Lemma 5.2. Let $\alpha = 1/G(p)$. Then it follows that

$$\int_0^{+\infty} h(\alpha x) x^p \, \mathrm{d}x = \int_0^{+\infty} h(\Phi(x)) x^p \, \mathrm{d}x.$$

Set

$$g(t) = \int_t^{+\infty} \left(h(\alpha x) - h(\Phi(x))\right) x^p \, \mathrm{d}x.$$

Then by the definition of α we have $g(0) = 0$. Obviously $g(\infty) = 0$. We are able to analyse the sign of $h(\alpha x) - h(\Phi(x))$. Since $\Phi(x)/x$ is increasing, there exists $x_0 \in [0, +\infty]$ such that $\Phi(x) \leq \alpha x$ for $x < x_0$ and $\Phi(x) \geq \alpha x$ for $x > x_0$. Since h is decreasing we have $h(\alpha x) - h(\Phi(x)) \leq 0$ for $x < x_0$ and $h(\alpha x) - h(\Phi(x)) \geq 0$ for $x > x_0$. Therefore, we know the sign of $g'(t)$ and we can conclude that g is increasing on $[0, x_0]$ and decreasing on $[x_0, \infty)$. Since $g(0) = g(+\infty) = 0$, we deduce that $g \geq 0$ on $\mathbb{R}_+$.

The statement of the lemma follows by integration by parts. Indeed, taking $-1 < p \leq q$,

$$\begin{aligned}
\int_0^{+\infty} x^q h(\Phi(x)) \, \mathrm{d}x &= \int_0^{+\infty} x^p h(\Phi(x)) x^{q-p} \, \mathrm{d}x \\
&= (q-p) \int_0^{+\infty} x^p h(\Phi(x)) \int_0^x u^{q-p-1} \, \mathrm{d}u \, \mathrm{d}x \\
&= (q-p) \int_0^{+\infty} u^{q-p-1} \int_u^{+\infty} x^p h(\Phi(x)) \, \mathrm{d}x \, \mathrm{d}u \\
&\leq (q-p) \int_0^{+\infty} u^{q-p-1} \int_u^{+\infty} x^p h(\alpha x) \, \mathrm{d}x \, \mathrm{d}u \\
&= \int_0^{+\infty} h(\alpha x) x^q \, \mathrm{d}x = \frac{1}{\alpha^{q+1}} \int_0^{+\infty} h(x) x^q \, \mathrm{d}x,
\end{aligned}$$

which is equivalent to $G(q) \le G(p)$. □

5.3 Proposition. *Suppose* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *is log-concave with* $f(0) = 1$. *Then the function*

$$p \ \mapsto \ \left(\frac{\int_0^{+\infty} t^p f(t) \, \mathrm{d}t}{\Gamma(p+1)} \right)^{\frac{1}{p+1}}$$

is decreasing on $(-1, +\infty)$.

Proof. Let $h(t) = e^{-t}$. By log-concavity we have $f = e^{-\Phi}$, where Φ is convex on $\mathbb{R}_+$, and since $f(0) = 1$, we have $\Phi(0) = 0$. Clearly $f = h \circ \Phi$, therefore we can apply Lemma 5.2 and use the fact that

$$\int_0^{+\infty} e^{-x} x^p \, \mathrm{d}x = \Gamma(p+1).$$

□

We shall present now a crucial property of log-concave distributions which says that they have log-concave tails. It will be important in view of the next proposition, where we discuss the comparison of moments of random variables with log-concave tails.

5.4 Proposition. *If* $f : \mathbb{R} \to \mathbb{R}$ *is log-concave then it has log-concave tail, namely*

$$t \ \mapsto \ \int_t^{+\infty} f(x) \, \mathrm{d}x$$

is log-concave.

Proof. We define functions $g(x) = f(x)\mathbf{1}_{(t_1,\infty)}(x)$, $h(y) = f(y)\mathbf{1}_{(t_2,\infty)}(y)$ and $m(z) = f(z)\mathbf{1}_{(\lambda t_1 + (1-\lambda)t_2,\infty)}(z)$. Then log-concavity of f yields

$$m(\lambda x + (1-\lambda)y) \ge g(x)^{\lambda} h(y)^{1-\lambda}$$

and by the Prékopa-Leindler inequality, Theorem 3.6, we have

$$\int_{\lambda t_1 + (1-\lambda)t_2}^{+\infty} f(x) \, \mathrm{d}x \ge \left(\int_{t_1}^{+\infty} f(x) \, \mathrm{d}x \right)^{\lambda} \left(\int_{t_2}^{+\infty} f(x) \, \mathrm{d}x \right)^{1-\lambda}.$$

□

We state the reverse Hölder inequalities for positive random variables with log-concave tails. In particular, due to Proposition 5.4, it is valid for log-concave distributions as well.

5.5 Proposition. *Suppose Z is a positive random variable with log-concave tail, i.e. the function $f(t) = \mathbb{P}(Z > t)$ is log-concave. Let $\mathcal{E}$ be the exponential random variable with parameter 1. Then for $p > q > 0$ we have*

$$(\mathbb{E}Z^p)^{1/p} \leq \frac{(\mathbb{E}\mathcal{E}^p)^{1/p}}{(\mathbb{E}\mathcal{E}^q)^{1/q}} (\mathbb{E}Z^q)^{1/q}.$$

5.6 Remark. Note that $\mathbb{E}\mathcal{E}^p = \Gamma(p+1)$, so

$$\frac{(\mathbb{E}\mathcal{E}^p)^{1/p}}{(\mathbb{E}\mathcal{E}^q)^{1/q}} \leq C\frac{p}{q},$$

where C is a universal constant.

Proof. Define

$$G(p) = \frac{1}{\Gamma(p)} \int_0^{+\infty} f(x)x^{p-1}\,\mathrm{d}x = \frac{\mathbb{E}Z^p}{\mathbb{E}\mathcal{E}^p}.$$

By Proposition 5.3 we have

$$G(p)^{1/p} \leq G(q)^{1/q}.$$

□

The last proposition is a typical example of a reverse Hölder inequality. We deduce the so-called Khinchine type inequality for linear functionals.

5.7 Corollary. *Let $\theta \in S^{n-1}$, take $H = \theta^\perp$ and $H_+ = \{x \in \mathbb{R}^n,\ \langle x, \theta \rangle \geq 0\}$. Let*

$$\langle x, \theta \rangle_+ = \begin{cases} \langle x, \theta \rangle, & \textit{if } \langle x, \theta \rangle \geq 0 \\ 0 & \textit{otherwise} \end{cases}.$$

Then for every log-concave probability measure μ on $\mathbb{R}^n$

$$\left(\int_{\mathbb{R}^n} \langle x, \theta \rangle_+^p \,\mathrm{d}\mu(x) \right)^{1/p} \leq \frac{\Gamma(p+1)^{1/p}}{\Gamma(q+1)^{1/q}} \left(\int_{\mathbb{R}^n} \langle x, \theta \rangle_+^q \,\mathrm{d}\mu(x) \right)^{1/q} \tag{5.4}$$

for any $p \geq q > 0$.

In particular, it holds for a uniform measure μ on a convex body $K \subset \mathbb{R}^n$.

5.8 Remark. Since $|\langle x, \theta \rangle|^p = \langle x, \theta \rangle_+^p + \langle x, -\theta \rangle_+^p$, it is easy to see that inequality (5.4) holds true for the function $|\langle x, \theta \rangle|$ instead of $\langle x, \theta \rangle_+$.

Proof of Corollary 5.7. The function $\phi(x) = \langle x, \theta \rangle$ is affine, hence for every $u, v \geq 0$, and any $\lambda \in [0, 1]$

$$(1-\lambda)\{\langle x, \theta \rangle_+ \geq u\} + \lambda\{\langle x, \theta \rangle_+ \geq v\} \subset \{\langle x, \theta \rangle_+ \geq (1-\lambda)u + \lambda v\}.$$

By log-concavity of μ, we deduce that the function $t \mapsto f(t) = \mu(\{\langle x,\theta\rangle_+ \geq t\})$ is log-concave on $\mathbb{R}_+$. Let Z be a random variable with tail f. Then we have

$$\int_{\mathbb{R}^n} \langle x,\theta\rangle_+^p \,\mathrm{d}\mu(x) = p\int_0^{+\infty} t^{p-1}\mu(\{\langle x,\theta\rangle_+ \geq t\})\,\mathrm{d}t = \mathbb{E}Z^p$$

and the result follows from Proposition 5.5. □

We quickly explain why Lemma 5.2 is related to the study of the comparison of the volume of a convex body with the volume of its hyperplane sections.

5.9 Corollary. *Let K be a symmetric convex body and let $p>0$. Take $\theta \in S^{n-1}$ and $H=\theta^\perp$. Then*

$$\left(\frac{1}{|K|}\int_K |\langle x,\theta\rangle|^p \,\mathrm{d}x\right)^{1/p} \leq \frac{\mathrm{vol}_n K}{\mathrm{vol}_{n-1}(K\cap H)}\frac{n}{2}\left(\frac{n!}{(p+1)\dots(p+n)}\right)^{1/p}.$$

5.10 Remark. There is equality if K is a double cone

$$\mathrm{conv}\left\{\{x=(x_1,\dots,x_n)\in\mathbb{R}^n \mid x_1^2+\dots+x_{n-1}^2 \leq 1, x_n = 0\}, e_n, -e_n\right\}$$

and $H = e_n^\perp$.

5.11 Remark. If K is such that its inertia matrix is the identity, that is for all $\theta\in\mathbb{R}^n$

$$\frac{1}{|K|}\int_K |\langle x,\theta\rangle|^2\,\mathrm{d}x = |\theta|_2^2,$$

then taking $p=2$ in Corollary 5.9 yields

$$\frac{\mathrm{vol}_{n-1}(K\cap H)}{\mathrm{vol}_n K} \leq \sqrt{\frac{n^2}{2(n+1)(n+2)}}.$$

This type of inequality is related with the slicing problem.

Proof of Corollary 5.9. Since K is symmetric we have

$$\begin{aligned}\frac{1}{|K|}\int_K |\langle x,\theta\rangle|^p\,\mathrm{d}x &= \int_{\mathbb{R}} |t|^p \frac{\mathrm{vol}_{n-1}\{x\in K,\ \langle x,\theta\rangle = t\}}{\mathrm{vol}\,K}\,\mathrm{d}t\\ &= 2\int_0^{+\infty} t^p\frac{\mathrm{vol}_{n-1}\, K\cap(t\theta+H)}{\mathrm{vol}\,K}\,\mathrm{d}t.\end{aligned}$$

As a consequence of the Brunn-Minkowski inequality, we have seen in Theorem 3.17 that the function

$$f(t) = \frac{\mathrm{vol}_{n-1}\left(K\cap(t\theta+H)\right)}{\mathrm{vol}\,K}$$

is $\frac{1}{n-1}$ concave. Let $h(t) = (1-t)^{n-1}\mathbf{1}_{[0,1]}(t)$. Take

$$\Phi(x) = 1 - \left(\frac{f(x)}{f(0)}\right)^{\frac{1}{n-1}}.$$

Obviously Φ is convex on $\mathbb{R}$ and $\Phi(0) = 0$. Therefore, $x \mapsto \Phi(x)/x$ is increasing. Since $\max f = f(0)$, we have $\Phi(x) \in [0,1]$. By Lemma 5.2, we know that

$$G(p) = \left(\frac{\int_0^{+\infty} x^p \frac{f(x)}{f(0)}\, \mathrm{d}x}{\int_0^1 x^p(1-x)^{n-1}\, \mathrm{d}x}\right)^{\frac{1}{p+1}} = \left(\frac{\frac{1}{2}\frac{1}{|K|}\int_K |\langle x, \theta\rangle|^p\, \mathrm{d}x}{\frac{|K\cap H|}{|K|}\int_0^1 x^p(1-x)^{n-1}\, \mathrm{d}x}\right)^{\frac{1}{p+1}}$$

is decreasing for $p > -1$. Hence for all $p \geq 0$, $G(p) \leq G(0)$. Rewriting the inequality, we see that we are done. Indeed, the inequality $G(p) \leq G(0)$ is equivalent to

$$\left(\frac{\frac{1}{2|K|}\int_K |\langle x, \theta\rangle|^p\, \mathrm{d}x}{\frac{|K\cap H|}{|K|}B(p+1,n)}\right)^{\frac{1}{p+1}} \leq \frac{n}{2}\cdot\frac{|K|}{|K\cap H|}.$$

Therefore,

$$\begin{aligned}\left(\frac{1}{|K|}\int_K |\langle x, \theta\rangle|^p\, \mathrm{d}x\right)^{\frac{1}{p}} &\leq \left(\frac{2|K\cap H|}{|K|}B(p+1,n)\right)^{\frac{1}{p}}\left(\frac{n}{2}\cdot\frac{|K|}{|K\cap H|}\right)^{\frac{p+1}{p}} \\ &= \frac{n}{2}\frac{|K|}{|K\cap H|}(nB(p+1,n))^{1/p}.\end{aligned}$$

It suffices to notice that

$$nB(p+1,n) = n\frac{\Gamma(n)\Gamma(p+1)}{\Gamma(n+p+1)} = \frac{n!\Gamma(p+1)}{\Gamma(n+p+1)} = \frac{n!}{(p+1)\dots(p+n)}.$$

□

Now we conclude this section with the strongest form of generalisation of Berwald's inequality.

5.12 Theorem. *Let $f : [0,\infty) \to [0,\infty)$ be $1/n$-concave on its support. We define $H : [-1,\infty) \to \mathbb{R}_+$ by*

$$H(p) = \begin{cases} \dfrac{\int_0^{+\infty} t^p f(t)\, \mathrm{d}t}{B(p+1,n+1)} & p > -1 \\ f(0) & p = -1 \end{cases}.$$

Then H is log-concave on $[-1,+\infty)$.

Another proof of Theorem 5.1. We have

$$\frac{1}{|K|}\int_K \phi(x)^{p+1}\,\mathrm{d}x = (p+1)\int_0^{+\infty} t^p \mu(\{\phi \geq t\})\,\mathrm{d}t,$$

where $\mu(A) = |K \cap A|/|K|$. We have seen in (5.3) that the function $f(t) = \mu(\{\phi \geq t\})$ is $1/n$-concave, therefore by Theorem 5.12,

$$p \in [-1, +\infty) \mapsto H(p) = \binom{n+p+1}{n}\frac{1}{|K|}\int_K \phi(x)^{p+1}\,\mathrm{d}x$$

is log-concave. Take $-1 \leq p \leq q$. We can find $\lambda \in [0,1]$ such that $p = (1-\lambda)(-1) + \lambda q$, i.e. $\lambda = \frac{p+1}{q+1}$. By log-concavity of H we have

$$H(p) \geq H(-1)^{1-\lambda} H(q)^{\lambda},$$

and since $H(-1) = f(0) = 1$ we obtain $H(p)^{1/(p+1)} \geq H(q)^{1/(q+1)}$. □

5.13 Corollary. *Let f be a log-concave function on $\mathbb{R}_+$. Then the function $H : [-1, \infty) \to \mathbb{R}_+$ given by*

$$H(p) = \begin{cases} \dfrac{\int_0^{+\infty} t^p f(t)\,\mathrm{d}t}{\Gamma(p+1)} & p > -1 \\ f(0) & p = -1 \end{cases}$$

is also log-concave. Moreover, if $f(0) = 1$ then the function

$$p \;\mapsto\; \left(\frac{\int_0^{+\infty} t^p f(t)\,\mathrm{d}t}{\Gamma(p+1)}\right)^{\frac{1}{p+1}}$$

is decreasing.

Proof. Let $f = e^{-\phi}$, where ϕ is convex. The function

$$g(t) = \left(1 - \frac{\phi(tn)}{n}\right)_+^n$$

is $(1/n)$-concave, therefore by Theorem 5.12 the function

$$p \;\mapsto\; \frac{\int_0^{+\infty} t^p \left(1 - \frac{\phi(tn)}{n}\right)_+^n \mathrm{d}t}{B(p+1, n+1)} = \frac{\Gamma(p+n+2)}{\Gamma(p+1)\Gamma(n+1)}\frac{1}{n^{p+1}}\int_0^{+\infty} s^p \left(1 - \frac{\phi(s)}{n}\right)_+^n \mathrm{d}s$$

is log-concave for $p \in [-1, +\infty)$. Letting $n \to \infty$ gives the result.

To prove the other part it suffices to observe that for every log-concave function F such that $F(0) = 1$ the function $x \mapsto -\frac{\ln(F(x))}{x}$ is increasing. Consequently, we have recovered Proposition 5.3. □

Proof of Theorem 5.12. Since f is nonnegative, $1/n$-concave on its support and satisfies some integrability condition at infinity (so that H is defined at least at one point), we know that f is supported on a finite interval.

Step 1. Take $-1 < p < q < r$. We can find nonnegative parameters a, b such that for the function $g_{a,b} : \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$g_{a,b}(t) = a\left(1 - \frac{t}{b}\right)^n \mathbf{1}_{[0,b]}(t)$$

we have

$$\int_0^{+\infty} t^p g_{a,b}(t)\ \mathrm{d}t = \int_0^{+\infty} t^p f(t)\ \mathrm{d}t := m_p,$$

and

$$\int_0^{+\infty} t^q g_{a,b}(t)\ \mathrm{d}t = \int_0^{+\infty} t^q f(t)\ \mathrm{d}t := m_q.$$

Indeed, for any $s > -1$

$$\int_0^{+\infty} t^s g_{a,b}(t)\ \mathrm{d}t = ab^{s+1}B(s+1, n+1),$$

so the solution reads

$$b = \left(\frac{m_p}{m_q} \cdot \frac{B(q+1,n+1)}{B(p+1,n+1)}\right)^{\frac{1}{p-q}}, \quad a = \left(\frac{m_p^{q+1}}{m_q^{p+1}} \cdot \frac{B(q+1,n+1)^{p+1}}{B(p+1,n+1)^{q+1}}\right)^{\frac{1}{q-p}}.$$

Step 2. Denote by H_g the function H associated with $g_{a,b}$. Then $H_g(s) = ab^{s+1}$, so we have

$$H_g(q) = H_g(p)^{1-\lambda}H_g(r)^{\lambda},$$

whenever $(1-\lambda)p + \lambda r = q$. This means that we have equality in the special case of H_g.

Step 3. Set $h = g - f$. We will prove that

$$\int_0^{+\infty} t^r h(t)\ \mathrm{d}t \geq 0. \tag{5.5}$$

This will conclude the statement since for $(1-\lambda)p + \lambda r = q$

$$H(q) = H_g(q) = H_g(p)^{1-\lambda}H_g(r)^{\lambda} \geq H(p)^{1-\lambda}H(r)^{\lambda}.$$

Let

$$H_1(t) = \int_t^{+\infty} s^p h(s)\ \mathrm{d}s, \quad H_2(t) = \int_t^{+\infty} s^{q-p-1}H_1(s)\ \mathrm{d}s.$$

We have $\int_0^{+\infty} t^p h(t)\ \mathrm{d}t = 0$, thus $H_1(\infty) = H_1(0) = 0$. We observe that

$$\begin{aligned}
0 &= \int_0^{+\infty} t^q h(t)\ \mathrm{d}t = \int_0^{+\infty} t^{q-p} t^p h(t)\ \mathrm{d}t \\
&= -\int_0^{+\infty} t^{q-p} H_1'(t)\ \mathrm{d}t = (q-p)\int_0^{+\infty} t^{q-p-1} H_1(t)\ \mathrm{d}t \\
&= (q-p) H_2(0),
\end{aligned}$$

whence $H_2(\infty) = H_2(0) = 0$. Since $H_2'(t) = -t^{q-p-1} H_1(t)$, the function H_1 changes sign at least once (if not, then $H_2' \geq 0$ or $H_2' \leq 0$, and since $H_2(0) = H_2(\infty) = 0$, we have $H_2 \equiv 0$, but then $H_1 \equiv 0$, $h \equiv 0$ and there is nothing to do). Since H_1 changes sign at least once and $H_1(0) = H_1(\infty) = 0$, therefore H_1' changes sign at least twice. Since $H_1'(t) = -t^p h(t)$, we have that h changes sign at least twice. Moreover, $g^{1/n}$ is affine and $f^{1/n}$ is concave. Therefore, h changes sign exactly twice at points t_1 and t_2 and we have $a > f(0)$ and $b > \max \operatorname{supp} f$.

Now we can analyse the behaviour of our functions,

t	0		t_1				t_2		$+\infty$
$h(t)$		$+$	0		$-$		0	$+$	
$H_1'(t)$		$-$	0		$+$		0	$-$	
$H_1(t)$	0	↘		↗	0	↗		↘	0
$H_2'(t)$			$+$		0		$-$		
$H_2(t)$	0		↗				↘		0

.

Hence $H_2 \geq 0$. Therefore,

$$\begin{aligned}
\int_0^{+\infty} t^r h(t)\ \mathrm{d}t &= \int_0^{+\infty} t^{r-p} t^p h(t)\ \mathrm{d}t = -\int_0^{+\infty} t^{r-p} H_1'(t)\ \mathrm{d}t \\
&= (r-p)\int_0^{+\infty} t^{r-p-1} H_1(t)\ \mathrm{d}t = (r-p)\int_0^{+\infty} t^{r-q} t^{q-p-1} H_1(t)\ \mathrm{d}t \\
&= -(r-p)\int_0^{+\infty} t^{r-q} H_2'(t)\ \mathrm{d}t = (r-p)(r-q)\int_0^{+\infty} t^{r-q-1} H_2(t)\ \mathrm{d}t \geq 0.
\end{aligned}$$

This proves (5.5). □

5.2 Some concentration inequalities

With a view to extending Corollary 5.7 to a vector setting, we require new tools. Indeed, a function like a norm $\|\cdot\|$ is not concave but convex and Theorem 5.1 cannot be applied. In this setting, the reverse Hölder inequalities are based on some concentration inequalities of log-concave measures.

5.14 Lemma. *Let K be a symmetric convex set in $\mathbb{R}^n$ and let μ be a log-concave probability measure such that $\mu(K) = \theta > \frac{1}{2}$. Then*

$$\mu((tK)^c) \leq \theta \left(\frac{1-\theta}{\theta}\right)^{\frac{t+1}{2}} \leq \left(\frac{1-\theta}{\theta}\right)^{\frac{t}{2}}, \qquad t \geq 1.$$

Proof. We prove that for any $t \geq 1$

$$K^c \supset \frac{2}{t+1}(tK)^c + \frac{t-1}{t+1}K. \tag{5.6}$$

To this end, suppose that $y \in K$ and that $z \notin tK$. If there was $x := \frac{2}{t+1}z + \frac{t-1}{t+1}y \in K$, then we would have $\frac{1}{t}z = \frac{t+1}{2t}x - \frac{t-1}{2t}y \in K$ by convexity and symmetry of K, a contradiction. Hence (5.6) is proved. By log-concavity of μ, we get

$$1-\theta = \mu(K^c) \geq \left[\mu((tK)^c)\right]^{\frac{2}{t+1}} \left[\mu(K)\right]^{\frac{t-1}{t+1}} = \left[\mu((tK)^c)\right]^{\frac{2}{t+1}} \theta^{\frac{t-1}{t+1}}.$$

Rewriting the expression, we arrive at

$$\mu((tK)^c) \leq (1-\theta)^{\frac{t+1}{2}} \theta^{\frac{1-t}{2}} = \theta \left(\frac{1-\theta}{\theta}\right)^{\frac{t+1}{2}} = \left(\frac{1-\theta}{\theta}\right)^{\frac{t}{2}} \sqrt{\theta(1-\theta)} \leq \left(\frac{1-\theta}{\theta}\right)^{\frac{t}{2}}.$$

Observe that this inequality is meaningful only if $\theta > 1/2$ so that $(1-\theta)/\theta < 1$. □

5.3 Kahane Khinchine type inequalities

5.15 Proposition. *Let μ be a log-concave probability measure on $\mathbb{R}^n$ and let $\|\cdot\|$ be a norm. Then*

$$\mu\left(\left\{\|x\| \geq 4t \int \|x\| d\mu(x)\right\}\right) \leq e^{-t/2}, \quad t \geq 1.$$

Proof. Let $I = \int \|x\| d\mu(x)$. Then by Markov's inequality $\mu(\|x\| \geq 4I) \leq 1/4$. Let K be the symmetric convex body defined by $\{x : \|x\| \leq 4\,I\}$. Then $\mu(K) = \theta \geq \frac{3}{4}$ and $(tK)^c = \{\|x\| \geq 4\,t\,I\}$. We conclude from Lemma 5.14 that

$$\mu(\{\|x\| \geq 4\,t\,I\}) = \mu((tK)^c) \leq \left(\frac{1-\theta}{\theta}\right)^{\frac{t}{2}} \leq 3^{-t/2} \leq e^{-t/2}.$$

□

Such exponential decay of the tails is related to the following reverse Hölder inequality.

5.16 Proposition. *Let μ be a log-concave probability measure and $\|\cdot\|$ be a norm. Then for any $1 \le p \le q$ we have*

$$\left(\int \|x\|^q \,\mathrm{d}\mu(x)\right)^{1/q} \le 12\,\frac{p}{q}\left(\int \|x\|^p \,\mathrm{d}\mu(x)\right)^{\frac{1}{p}}.$$

Proof. Observe that by replacing $\|\cdot\|$ by its multiple, we can assume that

$$\int \|x\|^p \,\mathrm{d}\mu(x) = 1.$$

Therefore, by Markov's inequality, we have

$$\mu(\{\|x\| \ge 4\}) \le 4^{-p}.$$

Take $K = \{x : \|x\| \le 4\}$. Then $\mu(K) = \theta \ge 1 - 4^{-p} > \frac{1}{2}$ and $(tK)^c = \{x : \|x\| > 4t\}$. By Lemma 5.14, we have

$$\mu(\{\|x\| > 4t\}) \le \left(\frac{1-\theta}{\theta}\right)^{\frac{t}{2}} \le \left(\frac{4^{-p}}{1-4^{-p}}\right)^{\frac{t}{2}} \le e^{-tp/2} \quad \text{for } t \ge 1$$

since $4^p \ge e^p + 1$ for all $p \ge 1$. Using this inequality, we can write

$$\begin{aligned}
\int \|x\|^q \,\mathrm{d}\mu(x) &= q\int_0^4 t^{q-1}\mu(\{\|x\| \ge t\})\,\mathrm{d}t + q\int_4^{+\infty} t^{q-1}\mu(\{\|x\| \ge t\})\,\mathrm{d}t \\
&\le 4^q + 4^q q\int_1^{+\infty} s^{q-1}\mu(\{\|x\| \ge 4s\})\,\mathrm{d}s \\
&\le 4^q + 4^q q\int_0^{+\infty} s^{q-1}e^{-s\,p/2}\,\mathrm{d}s \\
&\le 4^q + \left(\frac{8}{p}\right)^q \Gamma(q+1).
\end{aligned}$$

Since for every $q \ge 1$, $\Gamma(q+1)^{1/q} \le q$, we deduce that for any $1 \le p \le q$,

$$\left(\int \|x\||^q \,\mathrm{d}\mu(x)\right)^{1/q} \le 4 + 8\,\frac{q}{p} \le 12\,\frac{q}{p}.$$

□

5.17 Remark. For $\theta \in S^{n-1}$, one can take $\|x\| = |\langle x, \theta\rangle|$ whose unit ball is the symmetric strip $\{x : |\langle x, \theta\rangle| \le 1\}$. In that case, Proposition 5.16 is, up to a universal constant, the same as the symmetric version of Corollary 5.7

5.4 Notes and comments

The starting point of this section is an old result of Berwald [15]. His paper is readable and it is not difficult to go over all the statements to see that Theorem 5.1 is nothing else but *Satz 7* from [15]. We followed a modern presentation of the results. Lemma 5.2 is taken from [76] and the reader may find there several other types of reverse Hölder inequalities. Moreover, paper [76] contains a beautiful presentation of the slicing problem for convex bodies, where relations between the volume of sections of a convex set with the moments of linear functionals are established, like Corollary 5.9. The strongest form of reverse Hölder inequalities, Theorem 5.12 and Corollary 5.13 appeared in [20], see also [31]. Corollary 5.7 is stated in [74].

The non-symmetric version of inequalities like those obtained in Corollary 5.9 and the functional versions are of interest in the study of geometry of convex bodies. It has been done by Makai-Martini [72] and Fradelizi [40, 41, 42].

The extension of Corollary 5.7 to a vector setting is known since the work of Borell [22]. Lemma 5.14 is taken from [22] and is known as Borell's lemma. Proposition 5.16, is also stated in [22]. It is usually referred to as the Kahane Khinchine type inequality because it implies the classical Kahane inequality, see [77]. However, Corollary 5.7 and Proposition 5.16 do not cover the case when p goes to zero. It had been an open problem for some time and took quite an effort. The problem for $p \geq 0$ was addressed by Latała [62]. Then the first named author proved the equivalence for negative exponents (see [54]), using a completely different approach — the so-called localization lemma. Latała's theorem as well as Guédon's result led to a strong result of Bobkov [16]. It is also worth to mention here that Latała's method could be used for the negative exponents, as it was shown in [70]. It has to be noticed that a more general statement than Lemma 5.14 has been established in [54]. Namely, we have the following theorem.

5.18 Theorem. *Let K be a symmetric convex body in $\mathbb{R}^n$ and let μ be log-concave probability measure. Then for any $t \geq 1$,*

$$\mu((tK)^c) \leq (1 - \mu(K))^{\frac{t+1}{2}}.$$

It is the key tool to prove Kahane Khinchine type inequalities for negative exponents [54]. More general concentration inequalities for level sets of functions instead of norms have been established by Bobkov [17], Bobkov-Nazarov [19] and Fradelizi [43].

6 Concentration of mass of a log-concave measure

6.1 The result

Let X be a random vector in $\mathbb{R}^n$, and define $\sigma_p(X)$ by

$$\sigma_p(X) = \sup_{\theta \in S^{n-1}} (\mathbb{E}| \langle X, \theta \rangle |^p)^{1/p}.$$

Our goal is to prove the following theorem.

6.1 Theorem. *There exists a constant C such that for any random vector X distributed according to a log-concave probability measure on $\mathbb{R}^n$, we have for all $p \geq 1$,*

$$(\mathbb{E}|X|_2^p)^{1/p} \leq C\left(\mathbb{E}|X|_2 + \sigma_p(X)\right). \tag{6.1}$$

Moreover, if X is such that for all $\theta \in S^{n-1}$, $\mathbb{E}\langle X, \theta\rangle^2 = 1$, then for any $t \geq 1$ we have

$$\mathbb{P}(|X|_2 \geq c_1\, t\sqrt{n}) \leq e^{-t\sqrt{n}}, \tag{6.2}$$

where c_1 is a universal constant.

Proof of the "moreover" part. Since X is distributed according to a log-concave probability, we get from Corollary 5.7 (or Proposition 5.16) that for all $p \geq 1$,

$$\sigma_p(X) = \sup_{\theta \in S^{n-1}} (\mathbb{E}| \langle X, \theta \rangle |^p)^{1/p} \leq C'\, p \sup_{\theta \in S^{n-1}} \left(\mathbb{E}| \langle X, \theta \rangle |^2\right)^{1/2},$$

where C' is a universal constant. Moreover, since for all $\theta \in S^{n-1}$, $\mathbb{E}| \langle X, \theta \rangle |^2 = 1$, we deduce that

$$\mathbb{E}|X|_2 \leq (\mathbb{E}|X|_2^2)^{1/2} = \left(\sum_{i=1}^{n} \mathbb{E}| \langle X, e_i \rangle |^2\right)^{1/2} = \sqrt{n}$$

and conclude from (6.1) that for all $p \geq 1$,

$$(\mathbb{E}|X|_2^p)^{1/p} \leq C\sqrt{n} + C'p.$$

For any $t \geq 1$ take $p = t\sqrt{n}$ and $c_1 = e(C + C')$ so that $c_1 t\sqrt{n} \geq e(C'p + C\sqrt{n}) \geq e(\mathbb{E}|X|_2^p)^{1/p}$ and by Markov's inequality

$$\mathbb{P}(|X|_2 \geq c_1\, t\sqrt{n}) \leq \mathbb{P}(|X|_2^p \geq eE|X|_2^p) \leq e^{-p} = e^{-t\sqrt{n}}.$$

$\square$

The proof of the main inequality (6.1) requires more work. The first step is a simple reduction to the symmetric case. Indeed, let X' be an independent copy of X. Then by the Minkowski and Jensen's inequalities,

$$\begin{aligned}(\mathbb{E}|X|_2^p)^{1/p} &\le \left(\mathbb{E}\big||X|_2 - \mathbb{E}|X'|_2\big|^p\right)^{1/p} + \mathbb{E}|X'|_2 \\ &\le \left(\mathbb{E}\big||X|_2 - |X'|_2\big|^p\right)^{1/p} + \mathbb{E}|X|_2 \le (\mathbb{E}|X - X'|_2^p)^{1/p} + \mathbb{E}|X|_2.\end{aligned}$$

Assuming that inequality (6.1) is proved in the symmetric case, we apply it to $X - X'$ (which is symmetric and log-concave, see Proposition 3.16) and get

$$(\mathbb{E}|X - X'|_2^p)^{1/p} \le C(\mathbb{E}|X - X'|_2 + \sigma_p(X - X')).$$

But $\mathbb{E}|X - X'|_2 \le 2\mathbb{E}|X|_2$ and $\sigma_p(X - X') \le 2\sigma_p(X)$. Therefore,

$$(\mathbb{E}|X|_2^p)^{1/p} \le 3\,C\ (\mathbb{E}|X|_2 + \sigma_p(X))\,.$$

This means that to conclude the proof of Theorem 6.1, we just need to prove inequality (6.1) for a log-concave symmetric random vector X in $\mathbb{R}^n$.

This is the purpose of the rest of this chapter. We start off by introducing Z_p-bodies.

6.2 The Z_p-bodies associated with a measure

6.2 Definition. *Let μ be a measure on $\mathbb{R}^n$. We define the convex set $Z_p(\mu)$ by its support function*

$$h_{Z_p(\mu)}(\theta) = \left(\int \langle x, \theta\rangle_+^p \,\mathrm{d}\mu(x)\right)^{\frac{1}{p}}, \qquad \theta \in \mathbb{R}^n.$$

where $\langle x, \theta\rangle_+ = \langle x, \theta\rangle$ if $\langle x, \theta\rangle > 0$ and 0 otherwise.

6.3 Remark. To justify Definition 6.2, recall that the support function of a convex body K is given by $h_K(u) = \sup_{x\in K}\langle x, u\rangle$. And it is well known that any function h satisfying $h(\lambda x) = \lambda h(x)$ and $h(x + y) \le h(x) + h(y)$ for any $\lambda \ge 0$ and any $x, y \in \mathbb{R}^n$ is the support function of a unique convex set.

6.4 Remark. Let g be a standard Gaussian $\mathcal{N}(0, 1)$ random variable and let G be a standard Gaussian $\mathcal{N}(0, \mathrm{Id})$ random vector in $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$, $\langle G, x\rangle \sim g|x|_2$, hence we have

$$|x|_2^p = \mathbb{E}\,\langle G, x\rangle_+^p\,\frac{1}{\mathbb{E}g_+^p}.$$

Therefore,

$$\int |x|_2^p \,\mathrm{d}\mu(x) = \mathbb{E}\int \langle G, x\rangle_+^p \,\mathrm{d}\mu(x)\frac{1}{\mathbb{E}g_+^p} = \mathbb{E}(h^p_{Z_p(\mu)}(G))\frac{1}{\mathbb{E}g_+^p}. \tag{6.3}$$

The next Lemma is crucial to understand the properties of the Z_p-bodies associated with a measure with a density with respect to the Lebesgue measure on $\mathbb{R}^n$.

6.5 Lemma. *Let μ be a measure on $\mathbb{R}^n$ with density $w:\mathbb{R}^n\to\mathbb{R}_+$. Given a subspace $F\subset\mathbb{R}^n$, let $\Pi_F(\mu)$ be the marginal of μ, i.e.*

$$\Pi_F\mu(y)=\int_{y+F^\perp} w(x)\,\mathrm{d}x.$$

Then $P_F(Z_p(\mu))=Z_p(\Pi_F\mu)$.

Moreover, if w is log-concave and $w(0)>0$, let for any $r>0$

$$K_r(w)=\left\{x\in\mathbb{R}^n,\ r\int_0^{+\infty} t^{r-1}w(tx)\,\mathrm{d}t\geq w(0)\right\}.$$

Then for any $p>0$

$$Z_p(\mu)=w(0)^{1/p}Z_p(K_{n+p}(w))=w(0)^{1/p}\,|K_{n+p}(w)|^{\frac{1}{n}+\frac{1}{p}}\,Z_p(\widetilde{K_{n+p}(w)}),\tag{6.4}$$

where $Z_p(K_{n+p}(w))$ is the Z_p-body associated with the measure of density $\mathbf{1}_{K_{n+p}(w)}$ and $\widetilde{K_{n+p}(w)}$ is the homothetic image of $K_{n+p}(w)$ of volume 1.

Proof. For $\theta\in F$ we have

$$\begin{aligned}
h_{P_F(Z_p(\mu))}(\theta) &= \sup_{x\in P_F(Z_p(\mu))}\langle x,\theta\rangle=\sup_{y\in Z_p(\mu)}\langle P_F(y),\theta\rangle=\sup_{y\in Z_p(\mu)}\langle y,P_F\theta\rangle=h_{Z_p(\mu)}(\theta)\\
&=\left(\int\langle x,\theta\rangle_+^p\,\mathrm{d}\mu(x)\right)^{1/p}\\
&=\left(\int_F\int_{F^\perp}\langle y+z,\theta\rangle_+^p\,w(y+z)\,\mathrm{d}z\,\mathrm{d}y\right)^{1/p}\\
&=\left(\int_F\langle y,\theta\rangle_+^p\left(\int_{F^\perp}w(y+z)\,\mathrm{d}z\right)\mathrm{d}y\right)^{1/p}\\
&=\left(\int_F\langle y,\theta\rangle_+^p\ \Pi_F\mu(y)\,\mathrm{d}y\right)^{1/p}=h_{Z_p(\Pi_F\mu)}(\theta).
\end{aligned}$$

By Theorem 3.23 we know that when $w:\mathbb{R}^n\to\mathbb{R}_+$ is a log-concave function not 0 almost everywhere with $w(0)>0$, then the function

$$\|x\|_{K_r(w)}=\left(r\int_0^{+\infty}t^{r-1}\frac{w(tx)}{w(0)}\,\mathrm{d}t\right)^{-\frac{1}{r}}$$

is a gauge on $\mathbb{R}^n$ (recall that it is meant that it satisfies the triangle inequality). Therefore, the set $K_r(w) = \{\|x\|_{K_r(w)} \le 1\}$ is a convex set containing the origin. The second part of the lemma follows by integration in polar coordinates. We have

$$\begin{aligned}
h^p_{Z_p(\mu)}(\theta) &= \int \langle x,\theta\rangle_+^p \, w(x)\, \mathrm{d}x = n|B_2^n| \int_0^{+\infty} \int_{S^{n-1}} t^{n+p-1} \langle z,\theta\rangle_+^p \, w(tz)\, \mathrm{d}\sigma(z)\, \mathrm{d}t \\
&= w(0)\frac{n}{n+p}|B_2^n| \int_{S^{n-1}} \frac{\langle z,\theta\rangle_+^p}{\|z\|_{K_{n+p}}^{n+p}}\, \mathrm{d}\sigma(z) \\
&= n|B_2^n|w(0) \int_0^{+\infty} \int_{S^{n-1}} t^{n+p-1} \langle z,\theta\rangle_+^p \, \mathbf{1}_{K_{n+p}}(tz)\, \mathrm{d}\sigma(z)\, \mathrm{d}t \\
&= w(0) \int \langle x,\theta\rangle_+^p \, \mathbf{1}_{K_{n+p}}(x)\mathrm{d}x \\
&= w(0) h^p_{Z_p(K_{n+p})}(\theta).
\end{aligned}$$

Moreover, by a change of variable

$$\begin{aligned}
h_{Z_p(K_{n+p})}(\theta) &= \left(\int_{K_{n+p}} \langle x,\theta\rangle_+^p \, \mathrm{d}x \right)^{1/p} = |K_{n+p}|^{\frac{1}{p}+\frac{1}{n}} \left(\int_{\widetilde{K_{n+p}}} \langle x,\theta\rangle_+^p \, \mathrm{d}x \right)^{1/p} \\
&= |K_{n+p}|^{\frac{1}{p}+\frac{1}{n}} h_{Z_p(\widetilde{K_{n+p}})}(\theta)
\end{aligned}$$

which means that $Z_p(K_{n+p}) = |K_{n+p}|^{\frac{1}{n}+\frac{1}{p}} Z_p(\widetilde{K_{n+p}})$. The meaning of equality (6.4) is that in the log-concave case, the Z_p-bodies associated with the measure μ are the same as the Z_p-bodies associated with a properly defined convex body. □

In view of Lemma 6.5, we notice that it is of importance to work with family of measures which are stable after taking the marginals. By Theorem 3.15, we know that indeed, the marginals of a log-concave measure remains log-concave. At this stage, it is of interest to know some geometric properties of Ball's bodies, $K_r(w)$.

6.6 Proposition. *Let $w : \mathbb{R}^n \to \mathbb{R}_+$ be an even log-concave function such that $w(0) > 0$. For any $r > 0$ let*

$$K_r(w) = \left\{ x : r \int_0^{+\infty} t^{r-1} w(tx)\, \mathrm{d}t \ge w(0) \right\}.$$

Then for any $0 < s \le t$

$$K_s(w) \subset K_t(w) \subset \frac{\Gamma(t+1)^{1/t}}{\Gamma(s+1)^{1/s}}\, K_s(w).$$

Proof. For any $x \in \mathbb{R}^n$, let f_x be the log-concave function defined on $\mathbb{R}^+$ by $f_x(t) = w(tx)/w(0)$. Then Proposition 5.3 gives the right hand side inclusion. Indeed, suppose $x \in K_t(w)$. Then

$$\left(\frac{s\int_0^{+\infty} y^{s-1} f_x(y)\,\mathrm{d}y}{\Gamma(s+1)}\right)^{\frac{1}{s}} \geq \left(\frac{t\int_0^{+\infty} y^{t-1} f_x(y)\,\mathrm{d}y}{\Gamma(t+1)}\right)^{\frac{1}{t}} \geq \frac{1}{\Gamma(t+1)^{1/t}}.$$

Let $x = \frac{\Gamma(t+1)^{1/t}}{\Gamma(s+1)^{1/s}}\tilde{x}$. It follows that

$$\begin{aligned} s\int_0^{+\infty} y^{s-1} f_{\tilde{x}}(y)\,\mathrm{d}y &= s\int_0^{+\infty} y^{s-1} f_x\left(\frac{\Gamma(t+1)^{1/t}}{\Gamma(s+1)^{1/s}}y\right)\mathrm{d}y \\ &= \frac{\Gamma(t+1)^{s/t}}{\Gamma(s+1)} s\int_0^{+\infty} y^{s-1} f_x(y)\,\mathrm{d}y \geq 1. \end{aligned}$$

Therefore, $\tilde{x} \in K_s(w)$ and $x \in \frac{\Gamma(t+1)^{1/t}}{\Gamma(s+1)^{1/s}} K_s(w)$.

The left hand side is a consequence of Hölder inequality. Indeed, since w is even and log-concave, f_x is decreasing and right-continuous on $\mathbb{R}^+$. As a result, we can define a positive random variable Y such that for every $t > 0$, $\mathbb{P}(Y > t) = f(t)$ so that for every $r > 0$, $\|x\|_{K_r(w)} = (\mathbb{E}Y^r)^{-1/r}$. □

Since we have understood that in the case of a log-concave measure, the Z_p-bodies are the same as the Z_p-bodies of a properly defined convex set containing the origin, we investigate the properties of the Z_p-bodies in this particular case.

6.7 Proposition. *Let K be a symmetric convex body in $\mathbb{R}^n$ such that $|K| = 1$. Then for any $1 \leq p \leq q$,*

$$Z_p(K) \subset Z_q(K) \subset C\,\frac{q}{p}\; Z_p(K).$$

Moreover for any $p \geq n$,

$$Z_p(K) \supset cK \quad \text{and} \quad c \leq |Z_n(K)|^{1/n} \leq 1, \tag{6.5}$$

where c and C are positive universal constant.

Proof. The first inclusion follows from Corollary 5.7 (or Proposition 5.16). Now we prove the second part. Observe that for any p, and for every $\theta \in \mathbb{R}^n$

$$h^p_{Z_p(K)}(\theta) = \int_K \langle x,\theta\rangle_+^p\,\mathrm{d}x = p\int_0^{+\infty} t^{p-1} f(t)\,\mathrm{d}t,$$

where $f(t) = |\{x \in K : \langle x,\theta\rangle \geq t\}|$ is a $1/n$-concave function on $(0,+\infty)$. This follows from the Brunn-Minkowski inequality, see the proof of (5.3). We know from Theorem

5.12 that the function $H : [0,\infty) \to \mathbb{R}_+$ defined by

$$H(p) = \begin{cases} \dfrac{\int_0^{+\infty} t^{p-1} f(t)\,\mathrm{d}t}{B(p,n+1)} & p > 0 \\ f(0) & p = 0 \end{cases}$$

is log-concave on $[0,+\infty)$. Since $H(0) = f(0) = |K \cap \{\langle x,\theta\rangle \geq 0\}|$, we have by symmetry of K, $H(0) = 1/2$ and deduce that for any $p \leq q$,

$$2H(p) = \frac{H(p)}{H(0)} \geq \left(\frac{H(q)}{H(0)}\right)^{p/q} = (2H(q))^{p/q}. \tag{6.6}$$

We get

$$h_{Z_p(K)}(\theta) = \left(\frac{\Gamma(p+1)\Gamma(n+1)}{\Gamma(n+p+1)}\right)^{1/p} H(p)^{1/p}.$$

We conclude from (6.6) that for any $p \leq q$,

$$h_{Z_q(K)}(\theta) \leq \left(\frac{\Gamma(q+1)\Gamma(n+1)}{2\Gamma(q+n+1)}\right)^{1/q} \left(\frac{2\Gamma(p+n+1)}{\Gamma(p+1)\Gamma(n+1)}\right)^{1/p} h_{Z_p(K)}(\theta).$$

Since $|K| = 1$, we have

$$\lim_{q\to+\infty} h_{Z_q(K)}(\theta) = \max_{x\in K} |\langle x,\theta\rangle| = h_K(\theta)$$

and by properties of the Gamma function the first term tends to one, so we get for any $p \geq n$

$$h_{Z_p(K)}(\theta) \geq \left(\frac{\Gamma(p+1)\Gamma(n+1)}{2\Gamma(p+n+1)}\right)^{1/p} h_K(\theta) \geq \left(\frac{\Gamma(p+1)\Gamma(p+1)}{2\Gamma(p+p+1)}\right)^{1/p} h_K(\theta) \geq c\, h_K(\theta),$$

where c is a universal constant. □

6.8 Corollary. *Let w be an even log-concave density of a probability measure μ in $\mathbb{R}^n$. Then*

$$\frac{c}{w(0)^{1/n}} \leq |Z_n(\mu)|^{1/n} \leq \frac{C}{w(0)^{1/n}},$$

where c, C are absolute constants.

Proof. From (6.4),

$$Z_n(\mu) = w(0)^{1/n} Z_n(K_{2n}) = w(0)^{1/n} |K_{2n}|^{2/n} Z_n(\widetilde{K_{2n}}),$$

where $K_{2n} = K_{2n}(w)$ is a symmetric convex body in $\mathbb{R}^n$ and $\widetilde{K_{2n}}$ is its homothetic image of volume 1. We deduce from Proposition 6.7 that there is a universal constant c such that $c \leq |Z_n(\widetilde{K_{2n}})|^{1/n} \leq 1$. Therefore,

$$c\, w(0)^{1/n} |K_{2n}|^{2/n} \leq |Z_n(\mu)|^{1/n} \leq w(0)^{1/n} |K_{2n}|^{2/n}. \tag{6.7}$$

From Proposition 6.6,

$$|K_n|^{1/n} \leq |K_{2n}|^{1/n} \leq \frac{\Gamma(2n+1)^{1/2n}}{\Gamma(n+1)^{1/n}} |K_n|^{1/n} \leq C |K_n|^{1/n}.$$

By definition of K_n, we get after integration in polar coordinates,

$$|K_n| = |B_2^n| \int_{S^{n-1}} \frac{1}{\|\theta\|_{K_n}^n} \, \mathrm{d}\sigma(\theta) = n |B_2^n| \int_{S^{n-1}} \int_0^{+\infty} t^{n-1} \frac{w(t\theta)}{w(0)} \, \mathrm{d}t \, \mathrm{d}\sigma(\theta) = \frac{1}{w(0)} \int w(x) \, \mathrm{d}x.$$

Since μ is a probability measure, we have

$$|K_n| = w(0)^{-1} \tag{6.8}$$

and conclude that $w(0)^{-1/n} \leq |K_{2n}|^{1/n} \leq C w(0)^{-1/n}$. Combining this estimate with (6.7) gives the conclusion. □

6.9 Corollary. *Let w be an even log-concave density of a probability measure μ in $\mathbb{R}^n$. Then*

$$\left(\int_{\widetilde{K_{n+2}}} |x|_2^2 \, \mathrm{d}x \right)^{1/2} \leq w(0)^{1/n} \left(\int |x|_2^2 w(x) \, \mathrm{d}x \right)^{1/2} \leq C \left(\int_{\widetilde{K_{n+2}}} |x|_2^2 \, \mathrm{d}x \right)^{1/2},$$

where C is a universal constant. Moreover,

$$\left(\int_{\widetilde{K_{n+2}}} |x|_2^2 \, \mathrm{d}x \right)^{1/2} \geq \left(\int_{\widetilde{B_2^n}} |x|_2^2 \, \mathrm{d}x \right)^{1/2} = \sqrt{\frac{n}{n+2}} |B_2^n|^{-1/n} \geq c \sqrt{n}, \tag{6.9}$$

where c is a universal constant.

Proof. The proof is again based on (6.4). We use it with $p = 2$ and get that

$$\forall \theta \in \mathbb{R}^n, \; \int \langle x, \theta \rangle_+^2 \, w(x) \, \mathrm{d}x = w(0) \, |K_{n+2}|^{\frac{2}{n}+1} \int_{\widetilde{K_{n+2}}} \langle x, \theta \rangle_+^2 \, \mathrm{d}x. \tag{6.10}$$

From Proposition 6.6

$$|K_n|^{1/n} \leq |K_{n+2}|^{1/n} \leq \frac{\Gamma(n+3)^{\frac{1}{n+2}}}{\Gamma(n+1)^{\frac{1}{n}}} |K_n|^{1/n}.$$

Since $|K_n| = w(0)^{-1}$, see (6.8), we deduce that

$$w(0)^{-2/n} \le w(0)\,|K_{n+2}|^{\frac{2}{n}+1} \le \frac{(n+2)(n+1)}{\Gamma(1+n)^{2/n}}\; w(0)^{-2/n} \le C\, w(0)^{-2/n}$$

by properties of the Gamma function. To conclude, we observe that for any orthonormal basis $u_1, \ldots, u_n$ of $\mathbb{R}^n$, we have

$$\begin{aligned} w(0)^{2/n} \int |x|_2^2 w(x)\,\mathrm{d}x &= w(0)^{2/n} \sum_{i=1}^n \int \langle x, u_i\rangle^2\, w(x)\,\mathrm{d}x \\ &= w(0)^{2/n} \sum_{i=1}^n \int \langle x, u_i\rangle_+^2\, w(x)\,\mathrm{d}x + \int \langle x, -u_i\rangle_+^2\, w(x)\,\mathrm{d}x \end{aligned}$$

and we use (6.10).

The "moreover" part is in fact slightly more general. Let K be of volume 1. Then

$$\int_K |x|_2^2\,\mathrm{d}x = \int_{K\cap \widetilde{B_2^n}} |x|_2^2\,\mathrm{d}x + \int_{K\setminus \widetilde{B_2^n}} |x|_2^2\,\mathrm{d}x \ge \int_{\widetilde{B_2^n}} |x|_2^2\,\mathrm{d}x$$

since the Euclidean norm of any vector in $K \setminus \widetilde{B_2^n}$ is larger than for any vector in $\widetilde{B_2^n} \setminus K$ and $|K \setminus \widetilde{B_2^n}| = |\widetilde{B_2^n} \setminus K|$. □

6.3 The final step

Proof of inequality (6.1). Recall that to conclude the proof of Theorem 6.1 it is enough to prove that for any random vector X distributed according to a log-concave symmetric probability measure μ,

$$(\mathbb{E}|X|_2^p)^{1/p} \le C\,(\mathbb{E}|X|_2 + \sigma_p(X))\,. \tag{6.11}$$

Let k be the integer such that $p \le k < p+1$. Then $(\mathbb{E}|X|_2^p)^{1/p} \le (\mathbb{E}|X|_2^k)^{1/k}$ and by Corollary 5.7, $\sigma_k(X) \le \sigma_{p+1}(X) \le C\sigma_p(X)$. From (6.3) we have

$$(\mathbb{E}|X|_2^k)^{1/k} = \frac{\left(\mathbb{E}(h_{Z_k}^k(G))\right)^{1/k}}{\left(\mathbb{E}g_+^k\right)^{1/k}} \le \frac{C}{\sqrt{k}}\,\left(\mathbb{E}(h_{Z_k}^k(G))\right)^{1/k}, \tag{6.12}$$

where Z_k is associated with μ. Observe that $\sigma_k(X)$ is the smallest number b such that $Z_k \subset bB_2^n$. We split the discussion in two cases. Let c be a small enough constant.

If $k > \left(c\,\frac{\mathbb{E}h_{Z_k}(G)}{\sigma_k(X)}\right)^2$ we deduce from Theorem 4.20 that

$$\left(\mathbb{E}(h_{Z_k}^k(G))\right)^{1/k} \le C\,\sqrt{k}\,\sigma_k(X)$$

and (6.11) is proved.

If $k \le \left(c\, \frac{\mathbb{E} h_{Z_k}(G)}{\sigma_k(X)} \right)^2$ we deduce from Theorem 4.20 that

$$\left(\mathbb{E}(h^k_{Z_k}(G))\right)^{1/k} \le C\, \mathbb{E}(h_{Z_k}(G)). \tag{6.13}$$

Moreover, from Dvoretzky's Theorem, see Theorem 4.18, we get that the set of subspaces $E \in \mathcal{G}_{n,k}$ such that

$$\frac{1}{2} \frac{\mathbb{E} h_{Z_k}(G)}{\mathbb{E}|G|_2} P_E B_2^n \;\subset\; P_E Z_k \;\subset\; \frac{3}{2} \frac{\mathbb{E} h_{Z_k}(G)}{\mathbb{E}|G|_2} P_E B_2^n$$

has a measure greater than $1 - 4\exp(-ck)$. Therefore

$$\frac{\mathbb{E} h_{Z_k}(G)}{\mathbb{E}|G|_2} \le 2 \left(\frac{|P_E Z_k|}{|B_2^k|} \right)^{1/k} \le C' \sqrt{k}\, |P_E Z_k|^{1/k} \tag{6.14}$$

since it is well known that $|B_2^k|^{1/k} \ge c/\sqrt{k}$. The Z_k-body is associated with the symmetric log concave measure μ, therefore Lemma 6.5 implies that $P_E(Z_k) = Z_k(\Pi_E \mu)$. We conclude from Corollary 6.8 that

$$|P_E(Z_k)|^{1/k} \le \frac{C}{(\Pi_E \mu(0))^{1/k}}. \tag{6.15}$$

Combining (6.12), (6.13), (6.14) and (6.15), and using the fact that $\mathbb{E}|G|_2 \le \sqrt{n}$, we get

$$(\mathbb{E}|X|_2^k)^{1/k} \le \frac{C\sqrt{n}}{(\Pi_E \mu(0))^{1/k}}. \tag{6.16}$$

Let $Y = P_E X$ then $\Pi_E(\mu)$ is the density associated with Y which is even and log-concave. Since E is of dimension k, we deduce from Corollary 6.9 that

$$(\Pi_E \mu(0))^{1/k}\ (\mathbb{E}|Y|_2^2)^{1/2} \ge C\sqrt{k}.$$

Therefore the set of subspaces $E \in \mathcal{G}_{n,k}$ such that

$$(\mathbb{E}|X|_2^k)^{1/k} \le C \sqrt{\frac{n}{k}}\, (\mathbb{E}|P_E X|_2^2)^{1/2} \tag{6.17}$$

has a measure greater than $1 - 4\exp(-ck)$. Rotational invariance of the Haar measure $\nu_{n,k}$ on $\mathcal{G}_{n,k}$ implies that for each fixed $\theta_0 \in S^{n-1}$,

$$\mathbb{E}_{\nu_{n,k}} |P_E \theta_0|_2^2 = \int_{S^{n-1}} |P_{E_0} \theta|_2^2 \,\mathrm{d}\sigma(\theta)$$

where E_0 is a fixed subspace in $\mathcal{G}_{n,k}$. We can choose $E_0 = \mathrm{span}[e_1, \dots, e_k]$, where e_i are the vectors coming from the canonical basis of $\mathbb{R}^n$. Since for every $i = 1\dots, n$, $\int_{S^{n-1}} \theta_i^2 \,\mathrm{d}\sigma(\theta) = \int_{S^{n-1}} \theta_1^2 \,\mathrm{d}\sigma(\theta)$ we get

$$\int_{S^{n-1}} |P_{E_0}\theta|_2^2 \,\mathrm{d}\sigma(\theta) = \sum_{i=1}^{k} \int_{S^{n-1}} \theta_i^2 \,\mathrm{d}\sigma(\theta) = k \int_{S^{n-1}} \theta_1^2 \,\mathrm{d}\sigma(\theta) = \frac{k}{n} \sum_{i=1}^{n} \int_{S^{n-1}} \theta_i^2 \,\mathrm{d}\sigma(\theta) = \frac{k}{n}.$$

Therefore,

$$\mathbb{E}\mathbb{E}_{\nu_{n,k}} |P_E X|_2^2 = \frac{k}{n} \mathbb{E}|X|_2^2$$

and the set of subspaces $E \in \mathcal{G}_{n,k}$ such that

$$\left(\mathbb{E}|P_E X|_2^2\right)^{1/2} \leq \sqrt{\frac{(e^c/4)k}{n}} \left(\mathbb{E}|X|_2^2\right)^{1/2} \tag{6.18}$$

has a measure greater than $4\exp(-c)$. We can find a subspace E such that (6.17) and (6.18) hold true which proves that

$$(\mathbb{E}|X|_2^k)^{1/k} \leq C\, (\mathbb{E}|X|_2^2)^{1/2}.$$

By Proposition 5.16, we already know that $(\mathbb{E}|X|_2^2)^{1/2} \leq C\mathbb{E}|X|_2$ and this finishes the proof of (6.11). □

6.4 Notes, comments and further readings

Theorem 6.1 is due to Paouris [78]. It had a great influence on the theory of high dimensional convex bodies, as well as in the random matrix theory and the topics of probability in Banach spaces. In his paper Paouris assumed a log-concave measure to be in isotropic position. Theorem 6.1 is stated following [2] where the authors propose a new short proof of the result in particular avoiding the notion of Z_p-bodies associated with a measure. In [2], they propose this formulation because it corresponds to a probabilistic point of view. Indeed, it indicates that one can compare the strong moments and the weak moments of a log-concave random vector in a Hilbert space. It is conjectured in [63] that it still holds true in a general Banach space and some partial answers are given in [63], in particular for an unconditional log-concave measure. In this particular case, the "moreover" part of Theorem 6.1 was established by Bobkov and Nazarov [18]. To present the proof of Theorem 6.1, we have followed the original approach of Paouris except the fact that we have written all the formulas with a Gaussian random vector instead of the uniform measure on the sphere. Moreover, we have simplified the presentation by reducing the proof to the even log-concave setting. In this case, Proposition 6.7 and Corollary 6.8 are simpler to state and to prove. Their analogues in the case of a log-concave measure with barycentre at the origin are known and we refer to [79] for an extensive study of the Z_p-bodies. Paouris (see [79]) studied also the negative moments. This problem concerns small ball concentration.

In the log-concave setting, major progress has recently been made in the study of the concentration of mass in a Euclidean thin shell [59, 39, 60, 38, 56]. We refer to [55] for a short survey about the related open questions.

From a probabilistic point of view, it is worth noticing that Theorem 6.1 has been extended to the case of general convex measures [1].

The interested reader is encouraged to read the upcoming book [27].

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