An Upper Bound for Spherical Caps

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Abstract

We prove an useful upper bound for the measure of spherical caps.

Consider the uniformly distributed measure σ_{n-1} on the Euclidean unit sphere $S^{n-1} \subset \mathbb{R}^n$. On the sphere, as among only a handful other spaces, the isoperimetric problem is completely solved. This goes back to Lévy [Lé] and Schmidt [Sch] and states that caps have the minimal measure of a boundary among all sets with a fixed mass. For $\varepsilon \in [0, 1)$ and $\theta \in S^{n-1}$ the cap $C(\varepsilon, \theta)$, or shortly $C(\epsilon)$, is a set of points $x \in S^{n-1}$ for which $x \cdot \theta \geq \varepsilon$, where $\cdot$ stands for the standard scalar product in $\mathbb{R}^n$. See figure 1.

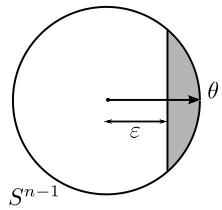

Figure 1: A cap $C(\varepsilon, \theta)$.

A few striking properties of the high-dimensional sphere are presented in [Ba, Lecture 1, 8]. In such considerations, we often need a good estimation of the measure of a cap. Following the method used in [Ba, Lemma 2.2], we extend its proof to the skipped case of large ε and get in an elementary way the desired bound.

Theorem. *For any* $\varepsilon \in [0, 1)$

$$\sigma_{n-1}\left(C(\varepsilon)\right) \leq e^{-n\varepsilon^2/2}.$$

Proof. In the case of small ε, for convenience, we repeat a beautiful argument used by Ball. Namely, for $\varepsilon \in [0, 1/\sqrt{2}]$ we have (see Figure 2)

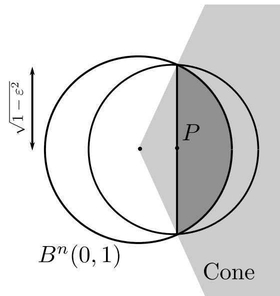

Figure 2: Small ε.

$$\begin{aligned}
\sigma_{n-1}\left(C(\varepsilon)\right) &= \frac{\mathrm{vol}_n\left(\mathrm{Cone} \cap B^n(0,1)\right)}{\mathrm{vol}_n\left(B^n(0,1)\right)} \\
&\leq \frac{\mathrm{vol}_n\left(B^n(P, \sqrt{1-\varepsilon^2})\right)}{\mathrm{vol}_n\left(B^n(0,1)\right)} \\
&= \sqrt{1-\varepsilon^2}^{\,n} \leq e^{-n\varepsilon^2/2}.
\end{aligned}$$

For $\varepsilon \in [1/\sqrt{2}, 1)$, it is enough to consider a different auxiliary ball which includes the set $\mathrm{Cone} \cap B^n(0,1)$, see Figure 3. We obtain

$$\sigma_{n-1}\left(C(\varepsilon)\right) \leq \frac{\mathrm{vol}_n\left(B^n(Q,r)\right)}{\mathrm{vol}_n\left(B^n(0,1)\right)} = r^n = \left(\frac{1}{2\varepsilon}\right)^n \leq e^{-n\varepsilon^2/2},$$

where the last inequality follows from the estimate

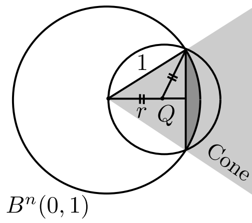

Figure 3: Large ε. By the congruence $\frac{1/2}{r} = \frac{\varepsilon}{1}$.

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$$e^{x^2/2} < 2x, \qquad \text{for } x \in [1/\sqrt{2}, 1].$$

Due to convexity, this is only to be checked at the boundary of our interval $[1/\sqrt{2}, 1]$, which reduces for both endpoints to the evident inequality $\sqrt{e} < 2$. □

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References

[Ba] K. Ball, *An elementary introduction to modern convex geometry. Flavors of geometry*, 1–58, Math. Sci. Res. Inst. Publ., **31**, Cambridge Univ. Press, Cambridge, 1997.

[Lé] P. Lévy, *Problèmes concrets d'analyse fonctionnelle*, Gauthier-Villars (1951).

[Sch] E. Schmidt. *Die Brunn-Minkowskische Ungleichung und ihr Spiegelbild sowie die isoperimetrische Eigenschaft der Kugel in der euklidischen und nichteuklidischen Geometrie*, Math. Nach. **1** (1948), 81–157.