A NOTE ON A BRUNN-MINKOWSKI INEQUALITY FOR THE GAUSSIAN MEASURE

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Abstract. We give the counter-examples related to a Gaussian Brunn-Minkowski inequality and the (B) conjecture.

1. Introduction and notation

Let γ_n be the standard Gaussian distribution on $\mathbb{R}^n$, i.e. the measure with the density

$$g_n(x) = \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/2},$$

where $|\cdot|$ stands for the standard Euclidean norm. A powerful tool in convex geometry is the Brunn-Minkowski inequality for Lebesgue measure (see [Sch] for more information). Concerning the Gaussian measure, the following question has recently been posed.

Question (R. Gardner and A. Zvavitch, [GZ])**.** *Let $0 < \lambda < 1$ and let A and B be closed convex sets in $\mathbb{R}^n$ such that $o \in A \cap B$. Is it true that*

$$\text{(GBM)} \qquad \gamma_n(\lambda A + (1-\lambda)B)^{1/n} \geq \lambda \gamma_n(A)^{1/n} + (1-\lambda)\gamma_n(B)^{1/n}?$$

A counter-example is given in this note. However, we believe that this question has an affirmative answer in the case of o-symmetric convex sets, i.e. the sets satisfying $K = -K$.

In [CFM] it is proved that for an o-symmetric convex set K in $\mathbb{R}^n$ the function

$$(1) \qquad \mathbb{R} \ni t \ \mapsto \gamma_n(e^t K),$$

is log-concave. This was conjectured by W. Banaszczyk and popularized by R. Latała [Lat]. It turns out that the (B) conjecture cannot be extended to the class of sets which are not necessarily o-symmetric yet contain the origin, as one of the sets provided in our counter-example shows.

As for the notation, we frequently use the function

$$T(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} \mathrm{d}t.$$

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2. Counter-examples

Now we construct the convex sets $A, B \subset \mathbb{R}^2$ containing the origin such that inequality (GBM) does not hold. Later on we show that for the set B the (B) conjecture is not true.

Fix $\alpha \in (0, \pi/2)$ and $\varepsilon > 0$. Take

$$A = \{(x,y) \in \mathbb{R}^2 \mid y \geq |x| \tan\alpha\},$$
$$B = B_\varepsilon = \{(x,y) \in \mathbb{R}^2 \mid y \geq |x| \tan\alpha - \varepsilon\} = A - (0,\varepsilon).$$

Clearly, A, B are convex and $0 \in A \cap B$. Moreover, from convexity of A we have $\lambda A + (1-\lambda)A = A$ and therefore

$$\lambda A + (1-\lambda)B = \lambda A + (1-\lambda)(A - (0,\varepsilon)) = A - (1-\lambda)(0,\varepsilon).$$

Observe that

$$\gamma_2(A) = \frac{1}{2} - \frac{\alpha}{\pi},$$
$$\gamma_2(B) = 2\int_0^{+\infty} T(x\tan\alpha - \varepsilon)\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,\mathrm{d}x,$$
$$\gamma_2(\lambda A + (1-\lambda)B) = 2\int_0^{+\infty} T(x\tan\alpha - \varepsilon(1-\lambda))\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,\mathrm{d}x$$

and that these expressions are analytic functions of ε. We will expand these functions in ε up to the order 2. Let

$$a_k = \int_0^{+\infty} T^{(k)}(x\tan\alpha)\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,\mathrm{d}x,$$

for $k = 0, 1, 2$, where $T^{(k)}$ is the k-th derivative of T (we adopt the standard notation $T^{(0)} = T$). We get

$$\gamma_2(A) = 2a_0,$$
$$\gamma_2(B) = 2a_0 - 2\varepsilon a_1 + \varepsilon^2 a_2 + o(\varepsilon^2),$$
$$\gamma_2(\lambda A + (1-\lambda)B) = 2a_0 - 2\varepsilon(1-\lambda)a_1 + \varepsilon^2(1-\lambda)^2 a_2 + o(\varepsilon^2).$$

Thus

$$\sqrt{\gamma_2(B)} = \sqrt{2a_0} - \frac{a_1}{\sqrt{2a_0}}\varepsilon + \left(\frac{a_2}{2\sqrt{2a_0}} - \frac{a_1^2}{2(2a_0)^{3/2}}\right)\varepsilon^2 + o(\varepsilon^2).$$

Taking $\varepsilon(1-\lambda)$ instead of ε we obtain

$$\sqrt{\gamma_2(\lambda A + (1-\lambda)B)} = \sqrt{2a_0} - \frac{a_1}{\sqrt{2a_0}}(1-\lambda)\varepsilon + \left(\frac{a_2}{2\sqrt{2a_0}} - \frac{a_1^2}{2(2a_0)^{3/2}}\right)(1-\lambda)^2\varepsilon^2 + o(\varepsilon^2).$$

Since

$$\sqrt{\gamma_2(\lambda A + (1-\lambda)B)} - \lambda\sqrt{\gamma_2(A)} - (1-\lambda)\sqrt{\gamma_2(B)} = -\lambda(1-\lambda)\frac{1}{2(2a_0)^{3/2}}(2a_0a_2 - a_1^2)\varepsilon^2 + o(\varepsilon^2),$$

we will have a counter-example if we find $\alpha \in (0, \pi/2)$ such that

$$2a_0a_2 - a_1^2 > 0.$$

Recall that $a_0 = \frac{1}{2}\gamma_2(A) = \frac{1}{2}\left(\frac{1}{2} - \frac{\alpha}{\pi}\right)$. The integrals that define the a_k's can be calculated. Namely,

$$\begin{aligned} a_1 &= \int_0^\infty T'(x\tan\alpha)\frac{e^{-x^2/2}}{\sqrt{2\pi}}\,\mathrm{d}x = -\frac{1}{\sqrt{2\pi}}\frac{1}{2}\int_{\mathbb{R}} e^{-(1+\tan^2\alpha)x^2/2}\frac{\mathrm{d}x}{\sqrt{2\pi}} \\ &= -\frac{1}{\sqrt{2\pi}}\frac{1}{2\sqrt{1+\tan^2\alpha}}, \\ a_2 &= \int_0^\infty T''(x\tan\alpha)\frac{e^{-x^2/2}}{\sqrt{2\pi}}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int_0^\infty (x\tan\alpha)e^{-(1+\tan^2\alpha)x^2/2}\frac{\mathrm{d}x}{\sqrt{2\pi}} \\ &= \frac{1}{2\pi}\frac{\tan\alpha}{1+\tan^2\alpha}. \end{aligned}$$

Therefore,

$$\begin{aligned} 2a_0a_2 - a_1^2 &= 2\left(\frac{1}{2}\left(\frac{1}{2} - \frac{\alpha}{\pi}\right)\cdot\frac{1}{2\pi}\frac{\tan\alpha}{1+\tan^2\alpha}\right) - \frac{1}{2\pi}\cdot\frac{1}{4(1+\tan^2\alpha)} \\ &= \frac{1}{8\pi}\frac{1}{1+\tan^2\alpha}\left(\tan\alpha\left(2 - \frac{4\alpha}{\pi}\right) - 1\right), \end{aligned}$$

which is positive for α close to $\pi/2$.

Now we turn our attention to the (B) conjecture. We are to check that for the set $B = B_\varepsilon$ the function $\mathbb{R} \ni t \mapsto \gamma_n(e^tB)$ is not log-concave, provided that ε is sufficiently small. Since

$$e^tB = \{(x,y) \in \mathbb{R}^2 \mid y \geq \tan\alpha|x| - \varepsilon e^t\}$$

we get

$$\begin{aligned} \ln\gamma_2(e^tB) &= \ln\left(2\int_0^\infty T(x\tan\alpha - e^t\varepsilon)\frac{e^{-x^2/2}}{\sqrt{2\pi}}\mathrm{d}x\right) \\ &= \ln\left(2\int_0^\infty T(x\tan\alpha)\frac{e^{-x^2/2}}{\sqrt{2\pi}}\mathrm{d}x\right) - \varepsilon e^t\frac{\int_0^\infty T'(x\tan\alpha)e^{-x^2/2}\mathrm{d}x}{\int_0^\infty T(x\tan\alpha)e^{-x^2/2}\mathrm{d}x} + o(\varepsilon). \end{aligned}$$

This produces the desired counter-example for sufficiently small ε as the function $t \mapsto \beta e^t$, where

$$\beta = -\frac{\int_0^\infty T'(x\tan\alpha)e^{-x^2/2}\mathrm{d}x}{\int_0^\infty T(x\tan\alpha)e^{-x^2/2}\mathrm{d}x} > 0,$$

is convex. □

Remark. The set B_ε which serves as a counter-example to the (B) conjecture in the nonsymmetric case works when the parameter $\alpha = 0$ as well (and ε is sufficiently small). Since B_ε is simply a halfspace in this case, it shows that symmetry of K is required for log-concavity of (1) even in the one-dimensional case.

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