On a convexity property of sections of the cross-polytope

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Abstract

We establish the log-concavity of the volume of central sections of dilations of the cross-polytope (the strong B-inequality for the cross-polytope and Lebesgue measure restricted to an arbitrary subspace).

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1 Introduction

The conjectured logarithmic Brunn-Minkowski inequality posed by Böröczky, Lutwak, Yang and Zhang in [3] can be equivalently stated as the following property of sections of the cube $B_\infty^n = [-1,1]^n$: for every subspace H of $\mathbb{R}^n$ the function

$$(t_1, \ldots, t_n) \mapsto \mathrm{vol}_H \left(\mathrm{diag}(e^{t_1}, \ldots, e^{t_n}) B_\infty^n \cap H\right)$$

is log-concave on $\mathbb{R}^n$. We explain this equivalence in Section 5. For a similar and other reformulations see the papers by Saroglou [8] and [9]. Here $\mathrm{diag}(e^{t_1}, \ldots, e^{t_n})$ denotes as usual the $n \times n$ diagonal matrix with e^{t_i} on the diagonal and vol_H denotes Lebesgue measure on H. In this note, we show that such a property holds for sections of the cross-polytope $B_1^n = \{x \in \mathbb{R}^n,\ \sum_{i=1}^n |x_i| \leq 1\}$.

Theorem 1. *Let H be a subspace of $\mathbb{R}^n$. Then the function*

$$(t_1, \ldots, t_n) \mapsto \mathrm{vol}_H \left(\mathrm{diag}(e^{t_1}, \ldots, e^{t_n}) B_1^n \cap H\right)$$

is log-concave on $\mathbb{R}^n$.

In other words, the so-called strong B-inequality holds for B_1^n and the (singular) measure being Lebesgue measure restricted to an arbitrary subspace of $\mathbb{R}^n$ (see the pioneering work [4] and see [9] for connections to the logarithmic Brunn-Minkowski inequality). We shall present in the sequel a simple example of a symmetric log-concave measure for which the strong B-property fails. Further examples of such measures have been recenlty found by Cordero-Erausquin and Rotem who have analysed in detail the strong B-property for centred Gaussian measures (see [5]).

It can be checked directly (and will also be clear from our proof) that the same holds true when B_1^n is replaced with B_2^n. We conjecture that the above theorem in fact holds for any ball $B_p^n = \{x \in \mathbb{R}^n,\ \sum_{i=1}^n |x_i|^p \leq 1\}$ put in place of B_1^n, $p > 1$.

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2 Proofs

2.1 Auxiliary results

The heart of our argument is the following probabilistic formula for volume of sections of dilations of the cross-polytope.

Lemma 2. *Let H be a codimension k subspace of $\mathbb{R}^n$. Let $u_1, \ldots, u_k$ be an orthonormal basis of the orthogonal complement of H and let $v_1, \ldots, v_n$ be the column vectors of the $k \times n$ matrix formed by taking $u_1, \ldots, u_k$ as its rows. Then for any positive numbers $a_1, \ldots, a_n$ we have*

$$\mathrm{vol}_H\left(\mathrm{diag}(a_1, \ldots, a_n) B_1^n \cap H\right) = \frac{2^{n-k}}{(n-k)! \cdot \pi^{k/2}} \left(\prod_{j=1}^n a_j\right) \mathbb{E}\left[\frac{1}{\sqrt{\det\left(\sum_{j=1}^n a_j^2 Y_j v_j v_j^T\right)}}\right],$$

where $Y_1, \ldots, Y_n$ are i.i.d. standard one sided exponential random variables.

Proof. The starting point is a well-known integral representation for volumes of sections: for an even, homogeneous and continuous function $N\colon \mathbb{R}^n \to [0, \infty)$ vanishing only at the origin and $p > 0$ we have

$$\Gamma(1 + (n-k)/p)\, \mathrm{vol}_{n-k}(\{x \in \mathbb{R}^n,\ N(x) \le 1\} \cap H) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^k} \int_{H(\epsilon)} e^{-N(x)^p} \mathrm{dvol}_n(x),$$

where H is, as in the assumptions of the lemma, a codimension k subspace of $\mathbb{R}^n$ whose orthogonal complement has an orthonormal basis $u_1, \ldots, u_k$ and

$$H(\epsilon) = \{x \in \mathbb{R}^n,\ |\langle x, u_j \rangle| \le \varepsilon/2,\ j = 1, \ldots, k\}.$$

This fact was probably first used in [7] and in this generality appeared for instance in [2] (Lemma 21). Its proof is based on Fubini's and Lebesgue's dominated convergence theorems. Using it for $p = 1$ and $N(x) = \sum a_i^{-1}|x_i|$, we get

$$(n-k)! \cdot \mathrm{vol}_H\left(\mathrm{diag}(a_1, \ldots, a_n) B_1^n \cap H\right) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^k} \int_{H(\varepsilon)} e^{-\sum a_i^{-1}|x_i|} \mathrm{d}x.$$

Let $X_1, \ldots, X_n$ be i.i.d. standard two-sided exponential random variables, that is with density $\frac{1}{2}e^{-|x|}$. Then the vector $(a_1X_1, \ldots, a_nX_n)$ has the density $\frac{1}{2^n \prod a_i} \exp(-\sum a_i^{-1}|x_i|)$, so

$$\begin{aligned}
&(n-k)! \cdot \mathrm{vol}_H\left(\mathrm{diag}(a_1, \ldots, a_n) B_1^n \cap H\right) \\
&\qquad = 2^n \left(\prod a_i\right) \lim_{\varepsilon \to 0} \varepsilon^{-k} \mathbb{P}\left((a_1X_1, \ldots, a_nX_n) \in H(\varepsilon)\right) \\
&\qquad = 2^n \left(\prod a_i\right) \lim_{\varepsilon \to 0} \varepsilon^{-k} \mathbb{P}\left(|\textstyle\sum_{i=1}^n a_i X_i u_{j,i}| \le \varepsilon/2,\ j = 1, \ldots, k\right).
\end{aligned}$$

Let us compute the probability above and then the limit. Recall the classical fact that the X_i are Gaussian mixtures (see also [6]). More preciesly, each X_i has the same distribution as the product $R_i \cdot G_i$ where the G_i are standard Gaussian random variables and R_i are i.i.d. positive random variables distributed as $\sqrt{2Y_i}$ with Y_i being i.i.d. standard one-sided exponentials (see a remark following Lemma 23 in [6]). If we condition on the R_i and introduce vectors $\tilde{u}_j = [a_i R_i u_{j,i}]_{i=1}^n$ we thus get

$$\mathbb{P}\left(|\textstyle\sum_{i=1}^n a_i X_i u_{j,i}| \le \varepsilon/2,\ j = 1, \ldots, k\right) = \mathbb{P}\left(|\langle G, \tilde{u}_j \rangle| \le \varepsilon/2,\ j = 1, \ldots, k\right),$$

where $G = (G_1, \ldots, G_n)$ is a standard Gaussian random vector. Let V be the subspace spanned by $\tilde{u}_1, \ldots, \tilde{u}_k$ and P_V the projection onto V. Then $G_V = P_V G$ is a standard Gaussian random vector on V. The above probability thus equals $\mathbb{P}(G_V \in \varepsilon K)$, where K is the subset of V given by $K = \{x \in \mathbb{R}^n \cap V, \ |\langle x, \tilde{u}_j\rangle| \le 1/2, \ j = 1, \ldots, k\}$, therefore it equals

$$\mathbb{P}(|\langle G, \tilde{u}_j\rangle| \le \varepsilon/2, \ j = 1, \ldots, k) = \mathbb{P}(G_V \in \varepsilon K) = \varepsilon^k (2\pi)^{-k/2} \operatorname{vol}_k(K) + o(\varepsilon^k).$$

We plug this back, use Lebesgue's dominated convergence theorem (notice that the function $\varepsilon^{-k}\mathbb{P}(G_V \in \varepsilon K)$ is majorised by $(2\pi)^{-k/2}\operatorname{vol}_k(K)$) and obtain

$$\begin{aligned}(n-k)! \cdot \operatorname{vol}_{n-k}\left(\{x \in \mathbb{R}^n, \ \sum a_i |x_i| \le 1\} \cap H\right) &= 2^n \left(\prod a_i\right) \lim_{\varepsilon \to 0} \varepsilon^{-k} \mathbb{E}_R \mathbb{P}(G_V \in \varepsilon K) \\ &= 2^n (2\pi)^{-k/2} \left(\prod a_i\right) \mathbb{E}_R \operatorname{vol}_k(K).\end{aligned}$$

We are almost done. It remains to recall an elementary fact that an intersection of exactly n strips in $\mathbb{R}^n$, say $\bigcap_{j=1}^n \{x \in \mathbb{R}^n, \ |\langle x, v_j\rangle| \le 1/2\}$ is an image of the cube $[-1/2, 1/2]^n$ under the linear map $(V^T)^{-1}$, where V is the matrix whose columns are the v_j (that is V maps the e_j onto v_j). Therefore the n-volume of the intersection is $\frac{1}{\det(V)}$. In other words, the volume is the reciprocal of the volume of the parallelotope $\{\sum t_i v_i, \ t_1, \ldots, t_n \in [0,1]\}$. Hence, in our case, $\operatorname{vol}_k(K)$ equals the volume of $\{\sum t_i \tilde{u}_i, \ t_1, \ldots, t_n \in [0,1]\}$. Thus,

$$\operatorname{vol}_k(K) = \frac{1}{\sqrt{\det(\tilde{U}^T \tilde{U})}},$$

where $\tilde{U}$ is the $n \times k$ matrix whose columns are the $\tilde{u}_j$. Noticing that the rows of $\tilde{U}$ are the vectors $a_i R_i v_i$ finishes the proof, since then

$$\frac{1}{\sqrt{\det(\tilde{U}^T \tilde{U})}} = \frac{1}{\sqrt{\det(\sum a_i R_i^2 v_i v_i^T)}}$$

and as mentioned earlier R_i has the same distribution as $\sqrt{2Y_i}$. □

We need the following standard lemma, whose proof can be found for example in [1] (see Lemma 1 and Lemma 2 (vi) therein).

Lemma 3. *Let $A_1, \ldots, A_n$ be $k \times k$ real symmetric positive semidefinite matrices. Then the function*

$$(x_1, \ldots, x_n) \mapsto \det\left(\sum_{i=1}^n x_i A_i\right)$$

is of the form

$$\sum_{1 \le j_1, \ldots, j_k \le n} b_{j_1, \ldots, j_k} x_{j_1} \cdot \ldots \cdot x_{j_k},$$

where $b_{j_1, \ldots, j_k} = D(A_{j_1}, \ldots, A_{j_k})$ is the mixed discriminant of $A_{j_1}, \ldots, A_{j_k}$. In particular, $b_{j_1, \ldots, j_k} \ge 0$.

Lemma 4. *Let $v_1, \ldots, v_n$ be vectors in $\mathbb{R}^k$. Then the function*

$$(t_1, \ldots, t_n) \mapsto \log\det(\sum e^{t_i} v_i v_i^T)$$

is convex on $\mathbb{R}^n$.

Proof. By Lemma 3, the function $f(t_1, \dots, t_n) = \det(\sum e^{t_i} v_i v_i^T)$ is of the form

$$f(t_1, \dots, t_n) = \sum_{1 \le j_1, \dots, j_k \le n} b_{j_1 \dots, j_k} e^{t_{j_1} + \dots + t_{j_k}},$$

for some nonnegative $b_{j_1 \dots, j_k}$. By Hölder's inequality,

$$f(\lambda s + (1-\lambda)t) \le f(s)^\lambda f(t)^{1-\lambda},$$

which finishes the proof. □

2.2 Proof of Theorem 1

Thanks to Lemma 2, it suffices to show that the function

$$\mathbb{E}\left[\det(\sum e^{t_i} Y_i v_i v_i^T)\right]^{-1/2} = \int_{(0,\infty)^n} \left[\det(\sum e^{t_i} y_i v_i v_i^T)\right]^{-1/2} e^{-\sum y_i} \mathrm{d}y.$$

is log-concave. We do the same change of variables $y_i = e^{s_i}$ as in [6] in the proof of the B-inequality for the exponential measure (Theorem 14). This gives

$$\int_{\mathbb{R}^n} \left[\det(\sum e^{t_i + s_i} v_i v_i^T)\right]^{-1/2} e^{-\sum (e^{s_i} - s_i)} \mathrm{d}y.$$

By Lemma 4 the integrand is a log-concave function of (s,t) on $\mathbb{R}^{2n}$ and by virtue of the Prékopa-Leindler inequality its marginal is also log-concave. □

2.3 Trouble with B_p^n for $1 < p < 2$

Let $1 < p < 2$. Since a random variable with the density proportional to $e^{-|x|^p}$ admits a representation as $R \cdot G$ for a standard Gaussian G and an independent positive random variable R (see [6]), repeating the same argument verbatim we can obtain an analogue of Lemma 2 for B_p^n in place of B_1^n. However, the final part of the proof of Theorem 1, where we change the variables $y_i = e^{s_i}$, will not lead to a log-concave integrand because $\log R$ is not log-concave for $1 < p < 2$ (see a discussion preceding Corollary 30 in [6]; see also [10]). (This is in contrast with the case when $R = \sqrt{2Y}$ with Y being standard exponential.) Currently we do not know how to remedy this inefficiency of our argument, but believe the theorem remains true for all p. On the other hand, the same remarks yield that Theorem 1 holds true for B_p^n with $0 < p < 1$.

3 Strong B-property

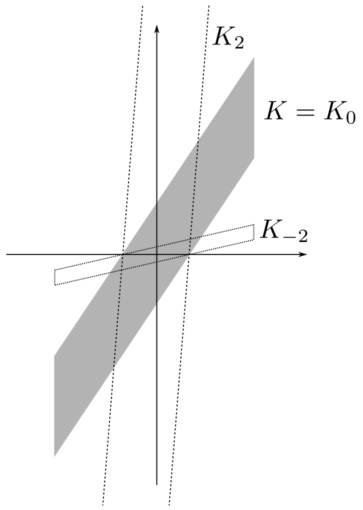
We say that a Borel measure μ on $\mathbb{R}^n$ satisfies the strong B-inequality if for every symmetric convex set K in $\mathbb{R}^n$ the function

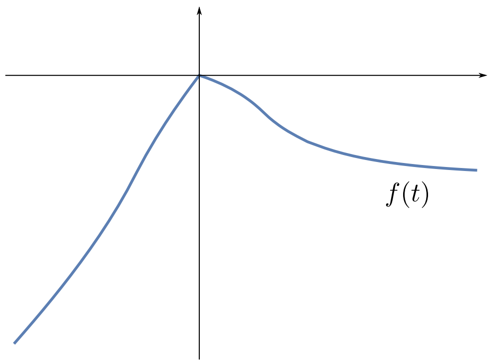
$$(t_1, \dots, t_n) \mapsto \mu(\mathrm{diag}(e^{t_1}, \dots, e^{t_n})K)$$

is log-concave on $\mathbb{R}^n$. Nontrivial examples of such measures include standard Gaussian measure and the product symmetric exponential measure (see [4] and [6]). We remark that it is not true that every symmetric log-concave measure satisfies the strong B-inequality (see also [5]). Take a uniform measure μ on the parallelogram

$$K = \mathrm{conv}\{(-1,-2), (-1,-1), (1,1), (1,2)\}$$

in $\mathbb{R}^2$. Let $K_t = \text{diag}(1, e^t)K$ and consider the function $f(t) = \log \mu(K_t) = \log \frac{|K_t \cap K|}{|K|}$. Clearly, $\max f = f(0) = 0$. Moreover, $\lim_{t \to -\infty} f(t) = -\infty$ (since $K_t \cap K$ converges to the interval $[-\frac{1}{3}, \frac{1}{3}] \times \{0\}$) and $\lim_{t \to \infty} f(t) > -\infty$ (since $K_t \cap K$ converges to the parallelogram $\text{conv}\{(-\frac{1}{3}, -\frac{2}{3}), (-\frac{1}{3}, 0), (\frac{1}{3}, 0), (\frac{1}{3}, \frac{2}{3})\}$). Such a function cannot be concave.





4 Another formula for volume of sections

Using the same probabilistic representation of the double-sided exponential distribution, we shall derive a complementary formula to the one from Lemma 2.

Lemma 5. *Let H be a k-dimensional subspace of $\mathbb{R}^n$ spanned by vectors $u_1, \ldots, u_k$ in $\mathbb{R}^n$ and let $v_1, \ldots, v_n$ be the column vectors of the $k \times n$ matrix formed by taking $u_1, \ldots, u_k$ as its rows. Then for any positive numbers $a_1, \ldots, a_n$ we have*

$$\begin{aligned}
&\text{vol}_H \left(\text{diag}(a_1, \ldots, a_n) B_1^n \cap H\right) \\
&\quad = \frac{2^k}{k! \cdot \pi^{(n-k)/2}} \sqrt{\det\left(\sum_{i=1}^n v_i v_i^T\right)} \mathbb{E}\left[\frac{1}{\sqrt{\prod_{i=1}^n Y_i}} \frac{1}{\sqrt{\det\left(\sum_{i=1}^n \frac{1}{Y_i a_i^2} v_i v_i^T\right)}}\right],
\end{aligned}$$

where $Y_1, \ldots, Y_n$ are i.i.d. standard one sided exponential random variables.

Proof. Let

$$K = \left\{ y \in \mathbb{R}^k, \ \sum_{i=1}^n a_i^{-1} |\langle y, v_i \rangle| \leq 1 \right\}.$$

Note that the set $\text{diag}(a_1, \ldots, a_n) B_1^n \cap H$ is the image of K under the linear injection $T : \mathbb{R}^k \to \mathbb{R}^n$ given by $Ty = [\langle y, v_i \rangle]_{i=1}^n$, $y \in \mathbb{R}^k$, whose image is H. Therefore,

$$\begin{aligned}
\text{vol}_H \left(\text{diag}(a_1, \ldots, a_n) B_1^n \cap H\right) &= \sqrt{\det(T^T T)} \, \text{vol}_k(K) \\
&= \sqrt{\det\left(\sum_{i=1}^n v_i v_i^T\right)} \, \text{vol}_k(K).
\end{aligned}$$

Let us develop the formula for the volume of K. Plainly, $\|y\|_K = \sum_{i=1}^n a_i^{-1} |\langle y, v_i\rangle|$, thus

$$\mathrm{vol}_k(K) = \frac{1}{k!}\int_{\mathbb{R}^k} e^{-\|y\|_K} \mathrm{d}y = \frac{1}{k!}\int_{\mathbb{R}^k} \prod_{i=1}^n e^{-a_i^{-1}|\langle y, v_i\rangle|}\mathrm{d}y.$$

Using as in the proof of Lemma 2 that a standard symmetric exponential random variable with density $\frac{1}{2}e^{-|x|}$ has the same distribution as $\sqrt{2Y}G$, where $Y \sim \mathrm{Exp}(1)$ and $G \sim N(0,1)$ are independent, we can write

$$\frac{1}{2}e^{-|x|} = \mathbb{E}\frac{1}{\sqrt{2\pi}\sqrt{2Y}}e^{-\frac{x^2}{4Y}}.$$

Taking i.i.d. copies $Y_1, \ldots, Y_n$ of Y, we obtain

$$\begin{aligned}
\mathrm{vol}_k(K) &= \frac{1}{k!}\int_{\mathbb{R}^k}\left(\mathbb{E}_Y \prod_{i=1}^n \frac{1}{\sqrt{\pi}\sqrt{Y_i}} e^{-\frac{\langle y, v_i\rangle^2}{4Y_i a_i^2}}\right)\mathrm{d}y \\
&= \frac{2^{k/2}}{k!\cdot\sqrt{\pi}^{n-k}}\mathbb{E}_Y\left[\frac{1}{\sqrt{\prod_{i=1}^n Y_i}}\int_{\mathbb{R}^k}\frac{1}{\sqrt{2\pi}^k}e^{-\frac{1}{2}\left\langle\left(\sum_{i=1}^n \frac{1}{2Y_i a_i^2} v_i v_i^T\right)y, y\right\rangle}\mathrm{d}y\right] \\
&= \frac{2^{k/2}}{k!\cdot\pi^{(n-k)/2}}\mathbb{E}_Y\left[\frac{1}{\sqrt{\prod_{i=1}^n Y_i}}\frac{1}{\sqrt{\det\left(\sum_{i=1}^n \frac{1}{2Y_i a_i^2} v_i v_i^T\right)}}\right].
\end{aligned}$$

Plugging this back to the formula for the volume of the section $\mathrm{diag}(a_1, \ldots, a_n)B_1^n \cap H$ finishes the proof. $\square$

Note that Lemma 5 uses k dimensional vectors, whereas Lemma 2 uses $n-k$ dimensional vectors, where k is the dimension of the section (subspace).

5 Connection to the log-Brunn-Minkowski inequality

Recall that for two origin symmetric convex bodies K and L in $\mathbb{R}^n$ and $\lambda \in [0,1]$, we define their *geometric mean* as

$$K^\lambda L^{1-\lambda} = \{x \in \mathbb{R}^n,\ \forall \theta \in \partial B_2^n\ \langle x, \theta\rangle \le h_K(\theta)^\lambda h_L(\theta)^{1-\lambda}\},$$

where h_K is the support functional of K, $h_K(\theta) = \sup_{y\in K}\langle y, \theta\rangle$ and similarly for L. Fix the dimension $n \ge 1$ and consider two statements

(i) *for every symmetric convex bodies K, L in $\mathbb{R}^n$ and $\lambda \in [0,1]$, we have*

$$\mathrm{vol}_n(K^\lambda L^{1-\lambda}) \ge \mathrm{vol}_n(K)^\lambda \,\mathrm{vol}_n(L)^{1-\lambda}$$

(ii) *for every $N \ge n$ and every n-dimensional subspace H of $\mathbb{R}^N$, the function*

$$F_H(t_1, \ldots, t_N) = \mathrm{vol}_H\left(\mathrm{diag}(e^{t_i})_{i=1}^N B_\infty^n \cap H\right)$$

is log-concave on $\mathbb{R}^N$.

Statement (i) is the conjectured log-Brunn-Minkowski inequality from [3], whereas statement (ii) is the aforementioned property of sections of the cube motivating our main result, Theorem 1. We shall now prove that they are equivalent (for a fixed $n \ge 1$).

Proof that (i) implies (ii). Let H be an n-dimensional subspace of $\mathbb{R}^N$, say H is given by vectors $v_1, \ldots, v_N \in \mathbb{R}^n$ as the image of $\mathbb{R}^n$ under the linear injection $T : \mathbb{R}^n \to \mathbb{R}^N$, $Ty = [\langle y, v_i \rangle]_{i=1}^N$, $y \in \mathbb{R}^n$. For $t \in \mathbb{R}^N$ define a convex symmetric set in $\mathbb{R}^n$,

$$K_t = \{x \in \mathbb{R}^n, \ \forall i \leq N \mid \langle x, v_i \rangle \mid \leq e^{t_i}\}. \tag{1}$$

Note that the image of K_t under T is the set $\mathrm{diag}(e^{t_i})_{i=1}^N B_\infty^n \cap H$. Therefore, we have $F_H(t) = \sqrt{\det(T^T T)}\, \mathrm{vol}_n(K_t)$. By the definition of the geometric mean, $K_s^\lambda K_t^{1-\lambda} \subset K_{\lambda s + (1-\lambda)t}$, so (i) gives the log-concavity of $t \mapsto \mathrm{vol}_n(K_t)$, hence F_H. □

Proof that (ii) implies (i). Let K and L be convex symmetric sets in $\mathbb{R}^n$. If we view their geometric mean $K^\lambda L^{1-\lambda}$ as the intersection over a countable dense subset of directions $v \in \partial B_2^n$ of the strips $\{x \in \mathbb{R}^n, \mid \langle x, v \rangle \mid \leq h_K(v)^\lambda h_L(v)^{1-\lambda}\}$, it is clear from continuity of measure that for a fixed $\varepsilon > 0$ there are directions $v_1, \ldots, v_N$ such that

$$\mathrm{vol}_n(K^\lambda L^{1-\lambda}) \geq \mathrm{vol}_n\{x \in \mathbb{R}^n, \forall i \leq N \mid \langle x, v_i \rangle \mid \leq h_K(v_i)^\lambda h_L(v_i)^{1-\lambda}\} - \varepsilon.$$

Let s_i and t_i be such that $e^{s_i} = h_K(v_i)$ and $e^{t_i} = h_L(v_i)$. Set H to be the image of $\mathbb{R}^n$ under $y \mapsto [\langle y, v_i \rangle]_{i=1}^N$. Using the notation of (1) we see that

$$\varepsilon + \mathrm{vol}_n(K^\lambda L^{1-\lambda}) \geq \mathrm{vol}_n(K_{\lambda s+(1-\lambda)t}) \geq \mathrm{vol}_n(K_s)^\lambda \,\mathrm{vol}_n(K_t)^{1-\lambda} \geq \mathrm{vol}_n(K)^\lambda \,\mathrm{vol}_n(L)^{1-\lambda},$$

where the second inequality follows from (ii) and the last inequality from the inclusions $K \subseteq K_s$ and $L \subseteq K_t$. If suffices to take $\varepsilon \to 0^+$.

□

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