21-260 Spring 2008
Homework # 9 Solutions

Section 6.4

(4) Applying $\mathcal{L}$ to both sides of the diff. eq., we have

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{\sin t\} + \mathcal{L}\{u_\pi(t)\sin(t-\pi)\}$$

$$= \mathcal{L}\{\sin t\} + e^{-\pi s}\mathcal{L}\{\sin t\}$$

$$= (1+e^{-\pi s})\mathcal{L}\{\sin t\}$$

$$= \frac{1+e^{-\pi s}}{s^2+1}$$

Then $\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0) = s^2\mathcal{L}\{y\}$

So $$s^2\mathcal{L}\{y\} + 4\mathcal{L}\{y\} = \frac{1+e^{-\pi s}}{s^2+1}$$

$$\Longrightarrow (s^2+4)\mathcal{L}\{y\} = \frac{1+e^{-\pi s}}{s^2+1}$$

$$\Longrightarrow \mathcal{L}\{y\} = \frac{1+e^{-\pi s}}{(s^2+4)(s^2+1)} \qquad (*)$$

Now $$\frac{1}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1}$$

$$\Longrightarrow 1 = (As+B)(s^2+1) + (Cs+D)(s^2+4)$$

$$1 = As^3 + As + Bs^2 + B + Cs^3 + 4Cs + Ds^2 + 4D$$

$$1 = (A+C)s^3 + (B+D)s^2 + (A+4C)s + (B+4D)$$

So we need $A+C=0$, $B+D=0$, $A+4C=0$, $B+4D=1$

So $A = -C$, but on the other hand, $A = -4C$. So this means $A = C = 0$. Then $B = -D$, so $B + 4D = -D + 4D = 3D = 1$

$\Rightarrow D = \frac{1}{3}$, and $B = -\frac{1}{3}$. Hence

$$\frac{1}{(s^2+4)(s^2+1)} = \frac{-1/3}{s^2+4} + \frac{1/3}{s^2+1}$$

Now we write $(*)$ as

$$\mathcal{L}\{y\} = \frac{1}{(s^2+4)(s^2+1)} + e^{-\pi s} \cdot \frac{1}{(s^2+4)(s^2+1)}$$; then since this

is $-\frac{1}{6} \cdot \frac{2}{s^2+4} + \frac{1}{3} \cdot \frac{1}{s^2+1}$

$$= -\frac{1}{6}\mathcal{L}\{\sin 2t\} + \frac{1}{3}\mathcal{L}\{\sin t\}$$

$$= \mathcal{L}\left\{\frac{1}{3}\sin t - \frac{1}{6}\sin 2t\right\},$$ so

$$\mathcal{L}\{y\} = \mathcal{L}\left\{\frac{1}{3}\sin t - \frac{1}{6}\sin 2t\right\} + \mathcal{L}\left\{u_\pi(t)\left[\frac{1}{3}\sin(t-\pi) - \frac{1}{6}\sin 2(t-\pi)\right]\right\}$$

$$\Rightarrow y(t) = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t + u_\pi(t)\left[\frac{1}{3}\underbrace{\sin(t-\pi)} - \frac{1}{6}\underbrace{\sin(2t-2\pi)}\right]$$

note this is the same as $\sin 2t$, and

this is just $-\sin t$

So $y(t) = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t + u_\pi(t)\left[-\frac{1}{3}\sin t - \frac{1}{6}\sin 2t\right]$

$$\Rightarrow y(t) = \frac{1}{3}(\sin t)(1 - u_\pi(t)) - \frac{1}{6}(\sin 2t)(1 + u_\pi(t))$$

7 $\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{u_{3\pi}\}$

$\Rightarrow s^2\mathcal{L}\{y\} - sy(0) - y'(0) + \mathcal{L}\{y\} = \dfrac{e^{-3\pi s}}{s}$

$\Rightarrow (s^2+1)\,\mathcal{L}\{y\} - s = \dfrac{e^{-3\pi s}}{s}$

$\Rightarrow (s^2+1)\,\mathcal{L}\{y\} = \dfrac{e^{-3\pi s}}{s} + s$

$$\Rightarrow \mathcal{L}\{y\} = \frac{e^{-3\pi s}}{s(s^2+1)} + \underbrace{\frac{s}{s^2+1}}_{\substack{\text{This is} \\ \mathcal{L}\{\cos t\}}}$$

Now $\dfrac{1}{s(s^2+1)} = \dfrac{A}{s} + \dfrac{Bs+C}{s^2+1}$

$\Rightarrow 1 = A(s^2+1) + (Bs+C)s$

$\Rightarrow 1 = As^2 + A + Bs^2 + Cs$

$\Rightarrow 1 = (A+B)s^2 + Cs + A$

So $A + B = 0 \Rightarrow A = -B$

$C = 0$

and $A = 1$, so $B = -1$

$\Rightarrow \dfrac{1}{s(s^2+1)} = \dfrac{1}{s} - \dfrac{s}{s^2+1}$

So $\mathcal{L}\{y\} = e^{-3\pi s}\left(\dfrac{1}{s} - \dfrac{s}{s^2+1}\right) + \mathcal{L}\{\cos t\}$

$$= e^{-3\pi s}\,\mathcal{L}\{1-\cos t\} + \mathcal{L}\{\cos t\}$$

$$= \mathcal{L}\{u_{3\pi}(t)[1-\cos(t-3\pi)]\} + \mathcal{L}\{\cos t\}$$

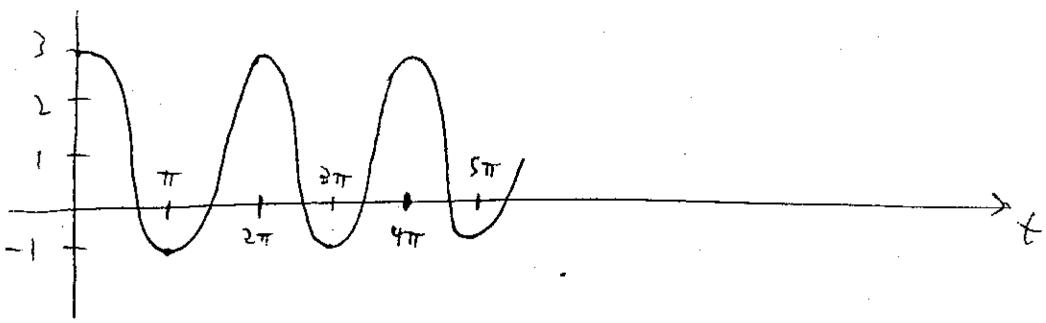
So $y(t) = u_{3\pi}(t)\,[1-\cos(t-3\pi)] + \cos t$

Now, $\cos(t-3\pi) = \cos(t-\pi) = -\cos t$, so

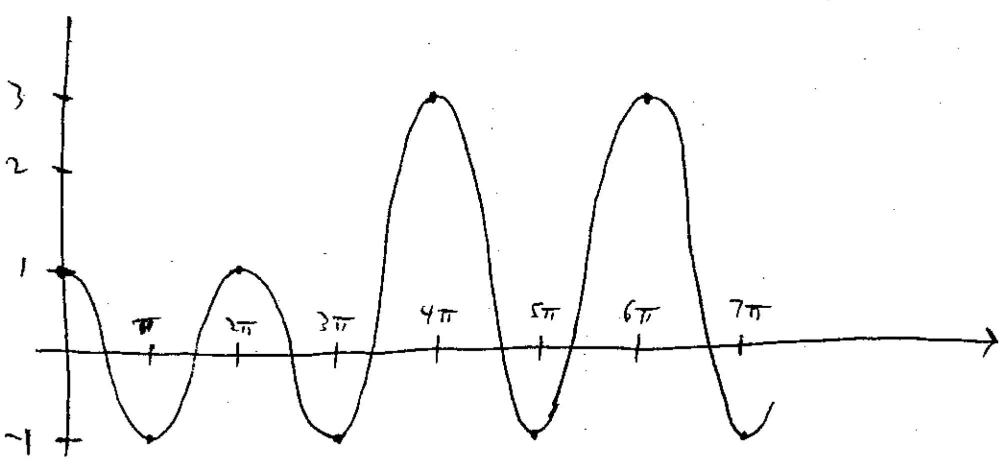
$$y(t) = u_{3\pi}(t)\,[1+\cos t] + \cos t$$

$$= \begin{cases} \cos t & \text{for } 0 \le t < 3\pi \\ 1 + 2\cos t & \text{for } t > 3\pi \end{cases}$$

Notice the function $g(t) = 1 + 2\cos t$ looks like this:



So $y(t)$ look-a like:



The external force which kicks in at $t = 3\pi$ causes the amplitude to double. And if you think about it, the fact that the solution agrees with $\cos t$ before that makes perfect sense, because the diff. eq. $y'' + y = 0$ applies before $t = 3\pi$, and we know that solutions to this (a.k.a. $y'' = -y$) are sines & cosines ... but look at the initial conditions: $y(0) = 1$ and $y'(0) = 0$. So it's fairly obvious (well ... once you see it) that we get the cosine function at first.

Then for $t \geq 3\pi$, the governing diff. eq. becomes $y'' + y = 1$, and if you were to solve the IVP

$$y'' + y = 1$$
$$y(3\pi) = -1$$
$$y'(3\pi) = 0$$

you would find that the solution is $y(t) = 1 + 2\cos t$

(10) We can write $g(t) = (1 - u_\pi(t))\sin t$, but the problem with this is that there is no formula handy for computing $\mathcal{L}\{g\}$ because its form doesn't correspond to anything in the left column of Table 6.2.1. A little algebraic manipulation will help:

$$g(t) = \sin t - u_\pi(t)\sin t = \sin t - u_\pi(t) \cdot \sin[(t-\pi) + \pi]$$

So ... well, wait a minute, that's stupid; let's do this:

$\sin(t-\pi) = -\sin t$, so $g(t) \doteq \sin t + u_\pi(t)\sin(t-\pi)$.

There we go. OK, now $\mathcal{L}\{y''\} + \mathcal{L}\{y'\} + \frac{5}{4}\mathcal{L}\{y\} = \mathcal{L}\{g\}$

$$= \mathcal{L}\{\sin t\} + \mathcal{L}\{u_\pi(t)\sin(t-\pi)\} = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1} = \frac{1+e^{-\pi s}}{s^2+1}$$

Then $\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0) = s^2\mathcal{L}\{y\}$

$\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0) = s\mathcal{L}\{y\}$

So we have $\left(s^2+s+\frac{5}{4}\right)\mathcal{L}\{y\} = \dfrac{1+e^{-\pi s}}{s^2+1}$

$$\Longrightarrow \mathcal{L}\{y\} = \frac{1+e^{-\pi s}}{\underbrace{\left(s^2+s+\frac{5}{4}\right)}(s^2+1)} \qquad (\star)$$

Now this quadratic factor is irreducible (i.e., not factor-able) because $1^2 - 4(1)\left(\frac{5}{4}\right) = 1-5 < 0$

Now we consider $\dfrac{1}{\left(s^2+s+\frac{5}{4}\right)(s^2+1)} = \dfrac{As+B}{s^2+s+\frac{5}{4}} + \dfrac{Cs+D}{s^2+1}$

$$\Longrightarrow 1 = (As+B)(s^2+1) + (Cs+D)\left(s^2+s+\frac{5}{4}\right)$$

$$1 = (A+C)s^3 + (B+C+D)s^2 + \left(A+\frac{5}{4}C+D\right)s + \left(B+\frac{5}{4}D\right)$$

Then we get the algebraic system

$$\left.\begin{array}{llllll} A & & +\ C & & = & 0 \\ & B & +\ C & +\ D & = & 0 \\ A & & +\frac{5}{4}C & +\ D & = & 0 \\ & B & & +\frac{5}{4}D & = & 1 \end{array}\right)$$

Leaving out the tedious details, the solution is

$A = \frac{16}{17}$, $B = \frac{12}{17}$, $C = -\frac{16}{17}$, $D = \frac{4}{17}$

So
$$\frac{1}{(s^2+s+\frac{5}{4})(s^2+1)} = \frac{4}{17}\left(\frac{4s+3}{s^2+s+\frac{5}{4}} + \frac{-4s+1}{s^2+1}\right) \qquad (*)$$

Now $s^2+s+\frac{5}{4} = (s^2+s+\frac{1}{4})+1 = (s+\frac{1}{2})^2+1$

Then $4s+3 = 4(s+\frac{1}{2})+1$, so $\frac{4s+3}{s^2+s+\frac{5}{4}}$

$$= \frac{4(s+\frac{1}{2})}{(s+\frac{1}{2})^2+1} + \frac{1}{(s+\frac{1}{2})^2+1} = \mathcal{L}\{4e^{-\frac{1}{2}t}\cos t + e^{-\frac{1}{2}t}\sin t\}$$

Next $\frac{-4s+1}{s^2+1} = \frac{-4s}{s^2+1} + \frac{1}{s^2+1} = \mathcal{L}\{-4\cos t + \sin t\}$

So $(*)$ is $\mathcal{L}\left\{\frac{4}{17}\left(e^{-\frac{1}{2}t}[4\cos t+\sin t] - 4\cos t + \sin t\right)\right\}$

and then from ($\star$) we get

$$y(t) = \frac{4}{17}\left(e^{-\frac{1}{2}t}[4\cos t+\sin t] - 4\cos t + \sin t\right)$$
$$+ u_\pi(t)\left[\frac{4}{17}e^{-\frac{1}{2}(t-\pi)}\left[4\cos(t-\pi)+\sin(t-\pi)\right] - 4\cos(t-\pi) + \sin(t-\pi)\right]$$

Section 7.5

(3) writing $x' = Ax$, with $A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$, we find the eigenvalues of A by considering $\det(A - rI) = \det\begin{bmatrix} 2-r & -1 \\ 3 & -2-r \end{bmatrix}$

$$= (2-r)(-2-r) + 3 = r^2 - 4 + 3 = r^2 - 1 = (r-1)(r+1) = 0$$

for $r_1 = 1$, $r_2 = -1$.

Considering $A - r_1 I = A - I = \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}$, and solving

$(A - I)\xi = 0$, or $\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 3 & -3 & 0 \end{array}\right]$, we get $\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\Rightarrow \xi_1 - \xi_2 = 0$, or $\xi_1 = \xi_2$. So $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, for instance, is a suitable eigenvector.

Then $A - r_2 I = A - (-1)I = A + I = \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}$,

and $(A + I)\xi = 0 \Rightarrow \left[\begin{array}{cc|c} 3 & -1 & 0 \\ 3 & -1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 3 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow$

$3\xi_1 - \xi_2 = 0 \Rightarrow \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ works as an eigenvector.

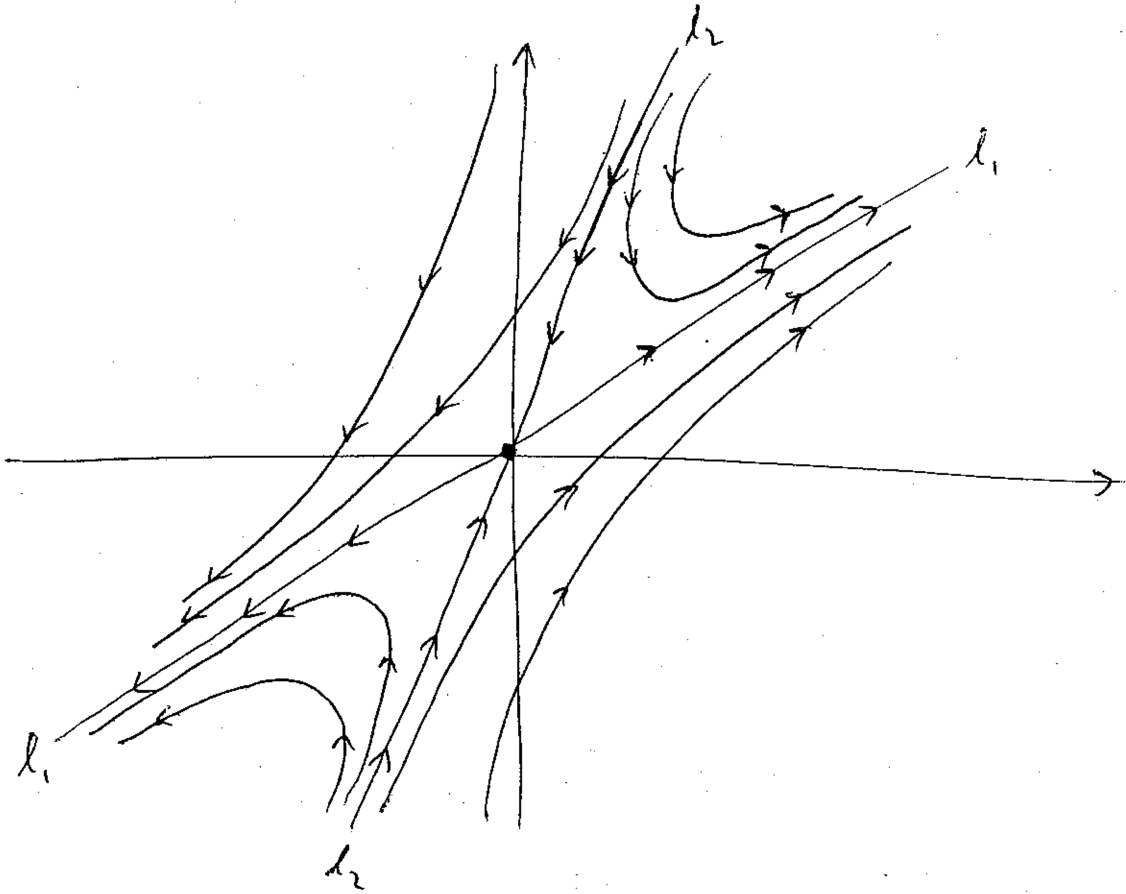
So the general solution to $x' = Ax$ is

$$x = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + \underbrace{c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}}$$

Now this term tends to 0 no matter the value of c_2

Letting $l_1 = \text{span}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $l_2 = \text{span}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, we see that solution trajectories approach l_1 unless $c_1 = 0$ (i.e., unless the solution starts out on l_2), in which case the trajectory approaches $(0,0)$.

So trajectories look like this:



(5) Write $x' = Ax$ with $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$, and $\det(A - rI)$

$$= \det \begin{bmatrix} -2-r & 1 \\ 1 & -2-r \end{bmatrix} = (-2-r)^2 - 1 = (r+2)^2 - 1 =$$

$$r^2 + 4r + 4 - 1 = r^2 + 4r + 3 = (r+3)(r+1) = 0$$ for

$r_1 = -3$ or $r_2 = -1$. (This tells us immediately that all solution trajectories approach the origin.)

Now $A - r_1 I = A - (-3)I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, so the system $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$,

or $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$, yields $\xi_1 + \xi_2 = 0$, and $\xi = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is a suitable eigenvector.

Then $A - r_2 I = A + I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ yields the system $\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$,

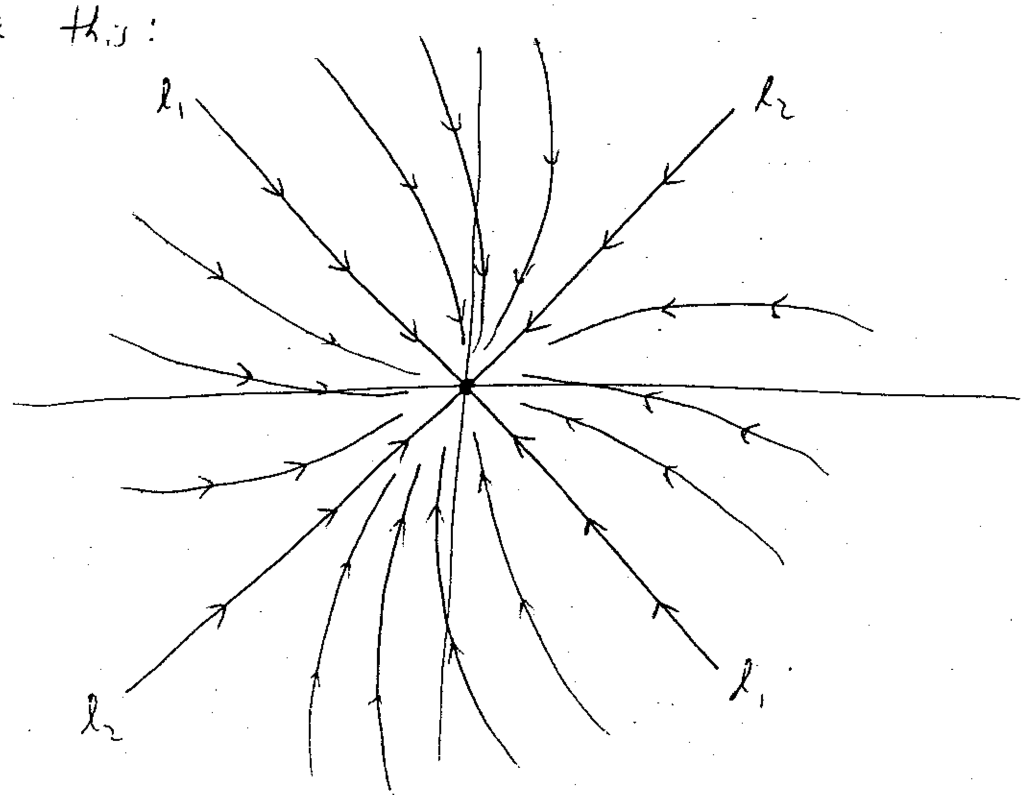
or $\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ with solution vectors ξ satisfying $\xi_2 - \xi_1 = 0$. So

$\xi = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a suitable eigenvector.

Solutions to $x' = Ax$ are therefore of the form

$$x(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

Let $l_1 = \text{span}\begin{bmatrix}1\\-1\end{bmatrix}$ and $l_2 = \text{span}\begin{bmatrix}1\\1\end{bmatrix}$, and solution trajectories look like this:



(6) Here $A = \begin{bmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{bmatrix}$, so $\det(A - rI) = \det\begin{bmatrix} \frac{5}{4} - r & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} - r \end{bmatrix}$

$= \left(r - \frac{5}{4}\right)^2 - \frac{9}{16} = r^2 - \frac{5}{2} r + \frac{25}{16} - \frac{9}{16} = r^2 - \frac{5}{2} r + 1$

$= 0 \iff 2r^2 - 5r + 2 = 0$; solutions are

$r = \dfrac{5 \pm \sqrt{25 - 16}}{4} = \dfrac{5 \pm 3}{4} \Rightarrow r_1 = 2,\ r_2 = \frac{1}{2}$

Now, $A - r_1 I = \begin{bmatrix} -\frac{3}{4} & \frac{3}{4} \\ \frac{3}{4} & -\frac{3}{4} \end{bmatrix}$, so $(A - r_1 I)\xi = 0$ implies

$\xi_1 = \xi_2$, so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ works as an eigenvector.

Then $A - r_2 I = A - \frac{1}{2} I = \begin{bmatrix} 3/4 & 3/4 \\ 3/4 & 3/4 \end{bmatrix}$, and $(A - \frac{1}{2}I)\xi = 0$

yields $\xi_1 = -\xi_2$. So $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is a suitable eigenvector corresponding

to $r_2 = \frac{1}{2}$.

So the general solution is $x(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{\frac{1}{2}t}$,

and unless the initial values are both 0, the individual solutions both grow, without bound, in magnitude. In fact, unless $c_1 = 0$ the solution trajectory always ends up in the first or third quadrant of the phase plane, moving roughly parallel to $l_1 = \text{span}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Let $l_2 = \text{span}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Trajectories look like this: