21-260 Spring 2008
Homework #8 Solutions

Section 6.2

(11) Applying $\mathcal{L}$ to both sides of the diff. eq., we get

$$\mathcal{L}\{y''-y'-6y\} = \mathcal{L}\{0\} = 0 \Rightarrow \mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 6\,\mathcal{L}\{y\} = 0$$

$$\Rightarrow s^2\mathcal{L}\{y\} - sy(0) - y'(0) - (s\,\mathcal{L}\{y\} - y(0)) - 6\,\mathcal{L}\{y\} = 0$$

$$\Rightarrow (s^2-s-6)\,\mathcal{L}\{y\} - sy(0) - y'(0) + y(0) = 0$$

$$\Rightarrow (s^2-s-6)\,\mathcal{L}\{y\} - s\cdot(1) - (-1) + 1 = 0$$

$$\Rightarrow (s^2-s-6)\,\mathcal{L}\{y\} - s + 2 = 0 \Rightarrow \mathcal{L}\{y\} = \frac{s-2}{s^2-s-6}$$

$$= \underbrace{\frac{s-2}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2}}$$ for some A, B.

Multiplying this equation through by $(s-3)(s+2)$, we have

$$s-2 = A(s+2) + B(s-3)$$

$$\Rightarrow s-2 = (A+B)s + (2A-3B)$$

arrrgh!

$$\Rightarrow \left.\begin{aligned} A+B &= 1 \\ 2A-3B &= -2 \end{aligned}\right] \Rightarrow A = \frac{1}{5},\ B = \frac{4}{5}$$

So $\mathcal{L}\{y\} = \frac{1}{5}\cdot\frac{1}{s-3} + \frac{4}{5}\cdot\frac{1}{s-(-2)}$

$\Rightarrow y(t) = \frac{1}{5}e^{3t} + \frac{4}{5}e^{-2t}$

(13) $\mathcal{L}\{y''-2y'+2y\} = \mathcal{L}\{0\}$

$\Rightarrow \mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = 0$

$\Rightarrow s^2\mathcal{L}\{y\} - sy(0) - y'(0) - 2(s\mathcal{L}\{y\} - y(0)) + 2\mathcal{L}\{y\} = 0$

$\Rightarrow (s^2-2s+2)\mathcal{L}\{y\} - 0 - 1 = 0$

$\Rightarrow \mathcal{L}\{y\} = \frac{1}{s^2-2s+2} = \frac{1}{(s^2-2s+1)+1}$

$= \frac{1}{(s-1)^2+1}$. Now see Item 9 in Table 6.2.1;

here we have $a=1$, $b=1$; so $y(t) = e^t \sin t$ is the solution to the IVP.

(17) $\mathcal{L}\{y^{(4)} - 4y''' + 6y'' - 4y' + y\} = \mathcal{L}\{y^{(4)}\} - 4\mathcal{L}\{y'''\}$
$+ 6\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + \mathcal{L}\{y\} = (s^4\mathcal{L}\{y\} - s^3y(0) - s^2y'(0)$
$- sy''(0) - y'''(0)) - 4(s^3\mathcal{L}\{y\} - s^2y(0) - sy'(0) - y''(0))$
$+ 6(s^2\mathcal{L}\{y\} - sy(0) - y'(0)) - 4(s\mathcal{L}\{y\} - y(0)) + \mathcal{L}\{y\}$
$= (s^4 - 4s^3 + 6s^2 - 4s + 1)\mathcal{L}\{y\} + (-s^3 + 4s^2 - 6s + 4)y(0)$

$$+ (-s^2 + 4s - 6)\, y'(0) + (-s + 4)\, y''(0) - y'''(0)$$

$$= (s^4 - 4s^3 + 6s^2 - 4s + 1)\, \mathcal{L}\{y\} + (-s^2 + 4s - 6) - 1 = 0$$

$$\Longrightarrow \mathcal{L}\{y\} = \frac{s^2 - 4s + 7}{s^4 - 4s^3 + 6s^2 - 4s + 1} = \frac{s^2 - 4s + 7}{(s-1)^4}$$

$$= \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3} + \frac{D}{(s-1)^4}$$

$$\Longrightarrow s^2 - 4s + 7 = A(s-1)^3 + B(s-1)^2 + C(s-1) + D$$

$$= A(s^3 - 3s^2 + 3s - 1) + B(s^2 - 2s + 1) + C(s-1) + D$$

$$= As^3 + (-3A + B)s^2 + (3A - 2B + C)s + (-A + B - C + D)$$

$$\Longrightarrow \left.\begin{aligned} A &= 0 \\ -3A + B &= 1 \\ 3A - 2B + C &= -4 \\ -A + B - C + D &= 7 \end{aligned}\right] \Longrightarrow \begin{aligned} &A = 0,\ B = 1, \\ &C = -2,\ D = 4 \end{aligned}$$

$$\text{So } \mathcal{L}\{y\} = \frac{1}{(s-1)^2} - \frac{2}{(s-1)^3} + \frac{4}{(s-1)^4}$$

NOTE THAT THIS LAST TERM IS $\frac{2}{3}\left(\frac{6}{(s-1)^4}\right)$

$$= \mathcal{L}\{te^t\} - \mathcal{L}\{t^2e^t\} + \mathcal{L}\{\tfrac{2}{3}t^3e^t\},$$

using Item 11 in Table 6.2.1

So $\mathcal{L}\{y\} = \mathcal{L}\{te^t - t^2e^t + \frac{2}{3}t^3e^t\}$

$\Rightarrow y(t) = e^t(t - t^2 + \frac{2}{3}t^3)$ is the unique solution to the IVP.

(22) $\mathcal{L}\{y'' - 2y' + 2y\} = \mathcal{L}\{e^{-t}\}$

Notice that since the left hand side of the diff. eq. is the same as in # 13, and the initial values of y and of y' are also the same. So the left hand side will (still) be $(s^2 - 2s + 2)\mathcal{L}\{y\} - 1$, but the right hand side is now $\mathcal{L}\{e^{-t}\} = \frac{1}{s+1}$.

So $\mathcal{L}\{y\} = \frac{1}{s^2-2s+2} \cdot \left[1 + \frac{1}{s+1}\right] = \frac{1}{s^2-2s+2} \cdot \frac{s+2}{s+1}$

$$= \frac{s+2}{(s^2-2s+2)(s+1)} = \frac{As+B}{s^2-2s+2} + \frac{C}{s+1}$$

$\Rightarrow s+2 = (As+B)(s+1) + C(s^2-2s+2)$

$\Rightarrow s+2 = As^2 + As + Bs + B + Cs^2 - 2Cs + 2C$

$\Rightarrow s+2 = (A+C)s^2 + (A+B-2C)s + (B+2C)$

$$\Rightarrow \left.\begin{array}{l} A + C = 0 \\ A + B - 2C = 1 \\ B + 2C = 2 \end{array}\right] \Rightarrow A = -\frac{1}{5}, \; B = \frac{8}{5}, \; C = \frac{1}{5}$$

So $\mathcal{L}\{y\} = \dfrac{-\frac{1}{5}s + \frac{8}{5}}{s^2 - 2s + 2} + \underbrace{\dfrac{1}{5} \cdot \dfrac{1}{s+1}}$

Now ... this is $\frac{1}{5}\mathcal{L}\{e^{-t}\}$, or $\mathcal{L}\{\frac{1}{5}e^{-t}\}$

But for the first term, we need a bit of algebraic razzmatazz:

write $$\frac{-\frac{1}{5}s + \frac{8}{5}}{(s-1)^2 + 1} = \frac{-\frac{1}{5}(s-1) - \frac{1}{5} + \frac{8}{5}}{(s-1)^2 + 1}$$

$$= -\frac{1}{5} \cdot \underbrace{\frac{s-1}{(s-1)^2 + 1}}_{\mathcal{L}\{e^t \cos t\}} + \frac{7}{5} \cdot \underbrace{\frac{1}{(s-1)^2 + 1}}_{\mathcal{L}\{e^t \sin t\}}$$

So $y(t) = -\frac{1}{5}e^t \cos t + \frac{7}{5}e^t \sin t + \frac{1}{5}e^{-t}$

(26) Writing the diff. eq. as $y'' + 4y = f(t)$, we have

$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{f\} \Rightarrow s^2\mathcal{L}\{y\} - sy(0) - y'(0)$

$+ 4\mathcal{L}\{y\} = \mathcal{L}\{f\} \Rightarrow (s^2 + 4)\mathcal{L}\{y\} = \mathcal{L}\{f\}.$

So now we need $\mathcal{L}\{f\} = \int_0^\infty f(t)\, e^{-st}\, dt$

$$= \int_0^1 t e^{-st}\, dt + \int_1^\infty e^{-st}\, dt$$

$$= \left[e^{-st}\left(-\frac{t}{s} - \frac{1}{s^2}\right)\right]_0^1 + \lim_{R\to\infty} \int_0^R e^{-st}\, dt$$

$$= -\frac{1}{s}e^{-s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} + \lim_{R\to\infty}\left[-\frac{1}{s}e^{-st}\right]_0^R$$

$$= -\frac{1}{s}e^{-s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} + \lim_{R\to\infty}\left(\underbrace{-\frac{1}{s}e^{-sR}} + \frac{1}{s}\right)$$

This tends to 0 as long as $s > 0$

$$= -\frac{1}{s}e^{-s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} + \frac{1}{s},$$

or $\dfrac{1-e^{-s}}{s} + \dfrac{1-e^{-s}}{s^2}$

So ... $\mathcal{L}\{y\} = \dfrac{1}{s^2+4}\left[\dfrac{1-e^{-s}}{s} + \dfrac{1-e^{-s}}{s^2}\right]$

Section 6.3

(8) write $t^2 - 2t + 2 = (t^2 - 2t + 1) + 1 = (t-1)^2 + 1$.

So $f(t) = u_1(t) \cdot [(t-1)^2 + 1]$, or $u_1(t)\, g(t-1)$ with

$g(t) = t^2 + 1$.

So $\mathcal{L}\{f\} = e^{-s} \mathcal{L}\{g\} = e^{-s} \mathcal{L}\{t^2 + 1\}$

$$= e^{-s}\left(\mathcal{L}\{t^2\} + \mathcal{L}\{1\}\right) = e^{-s}\left(\frac{2}{s^3} + \frac{1}{s}\right)$$

(10) $\mathcal{L}\{f\} = \mathcal{L}\{u_1\} + 2\mathcal{L}\{u_3\} - 6\mathcal{L}\{u_4\}$

$$= \frac{e^{-s}}{s} + \frac{2e^{-3s}}{s} - \frac{6e^{-4s}}{s}$$

$$= \frac{e^{-s} + 2e^{-3s} - 6e^{-4s}}{s}$$

(11) write $f(t) = (t - 2 - 1)\, u_2(t) - (t - 3 + 1)\, u_3(t)$

$$= (t-2)\, u_2(t) - (t-3)\, u_3(t) - u_2(t) - u_3(t)$$

So $\mathcal{L}\{f\} = \mathcal{L}\{(t-2)\cdot u_2\} - \mathcal{L}\{(t-3)\cdot u_3\} - \mathcal{L}\{u_2\} - \mathcal{L}\{u_3\}$

$$= e^{-2s}\mathcal{L}\{t\} - e^{-3s}\mathcal{L}\{t\} - \mathcal{L}\{u_2\} - \mathcal{L}\{u_3\}$$

$$= \frac{e^{-2s}}{s^2} - \frac{e^{-3s}}{s^2} - \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s}$$

or $\mathcal{L}\{f\} = \dfrac{e^{-2s} - e^{-3s}}{s^2} - \dfrac{e^{-2s} + e^{-3s}}{s}$

or $\mathcal{L}\{f\} = \dfrac{(1-s)e^{-2s} - (1+s)e^{-3s}}{s^2}$

(13) This conforms to ~~Item~~ 11 in Table 6.2.1, with $n = 3$ and $a = 2$, so $f(t) = t^3 e^{2t}$.

(16) First let's consider $\dfrac{1}{s^2-4} = \dfrac{1}{(s-2)(s+2)} = \dfrac{A}{s-2} + \dfrac{B}{s+2}$

we find $A = \frac{1}{4}$, $B = -\frac{1}{4}$ (details omitted), so

$$\frac{1}{s^2-4} = \frac{1}{4}\cdot\frac{1}{s-2} - \frac{1}{4}\cdot\frac{1}{s+2}$$

$$= \mathcal{L}\{\tfrac{1}{4}e^{2t}\} - \mathcal{L}\{\tfrac{1}{4}e^{-2t}\}$$

$$\Rightarrow \frac{1}{s^2-4} = \mathcal{L}\{\tfrac{1}{4}(e^{2t} - e^{-2t})\}$$

So now, view F as $e^{-2s} \cdot 2\,\mathcal{L}\{\frac{1}{4}(e^{2t} - e^{-2t})\}$

$$= e^{-2s}\,\mathcal{L}\{\tfrac{1}{2}(e^{2t} - e^{-2t})\}$$

$$= \mathcal{L}\{u_2(t)\cdot\tfrac{1}{2}(e^{2(t-2)} - e^{-2(t-2)})\}$$

using (3) on page 327.

If you are familiar with the hyperbolic (sine and cosine) functions, you'll recognize the function in braces as $u_2(t) \cdot \sinh[2(t-2)]$. If not, you can leave it as it is.

At any rate, $f(t) = \begin{cases} 0 & \text{for } 0 \leq t < 2 \\ \sinh[2(t-2)] & \text{for } t > 2 \end{cases}$

(18) Just using (2) on p. 326, we have

$f(t) = u_1(t) + u_2(t) - u_3(t) - u_4(t)$, the graph of which look-a like this: