21-260 Spring 2008
Homework #6 Solutions

Section 3.8

(6) Using $w = mg = kL$, we have $k = \frac{mg}{L} = \frac{(100 \text{ grams})(9.8 \text{ m/sec}^2)}{5 \text{ cm}}$

$$= \frac{(100 \text{ grams})(980 \text{ cm/sec}^2)}{5 \text{ cm}} = 19{,}600 \text{ grams/sec}^2$$

Now without damping we have $mu'' + ku = 0$, or

$100u'' + 19{,}600u = 0 \iff u'' + 196u = 0$, with general solution $u(t) = A\cos(14t) + B\sin(14t)$.

The initial conditions are $u(0) = 0$ and $u'(0) = 10$, so the first gives $A = 0$ so that our solution is of the form $B\sin(14t)$.

Therefore, $u'(t) = 14B\cos(14t)$, and $u'(0) = 10 \Rightarrow 14B = 10$ $\Rightarrow B = \frac{5}{7}$.

So $u(t) = \frac{5}{7}\sin(14t)$ gives the displacement at time t.

First return to equilibrium occurs when $14t = \pi \Rightarrow t = \frac{\pi}{14}$ ≈ 0.224 seconds ... real quick-like. So you can see that this load oscillates pretty rapidly.

(7) From $w = mg = kL$, we have $(20 \text{ grams})(980 \frac{\text{cm}}{\text{sec}^2}) = k \cdot (5 \text{ cm})$

$\Rightarrow k = 3920 \text{ grams/sec}^2$. Then $\gamma = 400 \text{ dyne-sec/cm}$, so we get the diff. eq. $20u'' + 400u' + 3920u = 0 \iff u'' + 20u' + 196u = 0$.

And in this case, the IVP is

$$\begin{cases} u'' + 20u' + 196u = 0 \\ u(0) = 2 \\ u'(0) = 0 \end{cases}$$

The characteristic eq. $r^2 + 20r + 196 = 0$ has roots

$$r = \frac{-20 \pm \sqrt{-384}}{2} = -10 \pm 9.798\,i$$

$\Rightarrow$ general solution is $u(t) = e^{-10t}\left(A\cos(9.798t) + B\sin(9.798t)\right)$

So we can see right away that the quasi-frequency is 9.798, and the quasi-period is $\frac{2\pi}{9.798} \approx 0.641$ seconds.

Now $u(0) = 2 \Rightarrow e^0\left(A\cos 0 + B\sin 0\right) = 2 \Rightarrow A = 2$.

Then
$$u'(t) = e^{-10t}\left(-2\cdot 9.798\sin(9.798t) + B\cdot 9.798\cos(9.798t)\right)$$
$$-10e^{-10t}\left(2\cos(9.798t) + B\sin(9.798t)\right),$$

so $u'(0) = 0 \rightarrow B\cdot 9.798 - 10\cdot 2 = 0 \rightarrow B = \frac{20}{9.798}$

≈ 2.041.

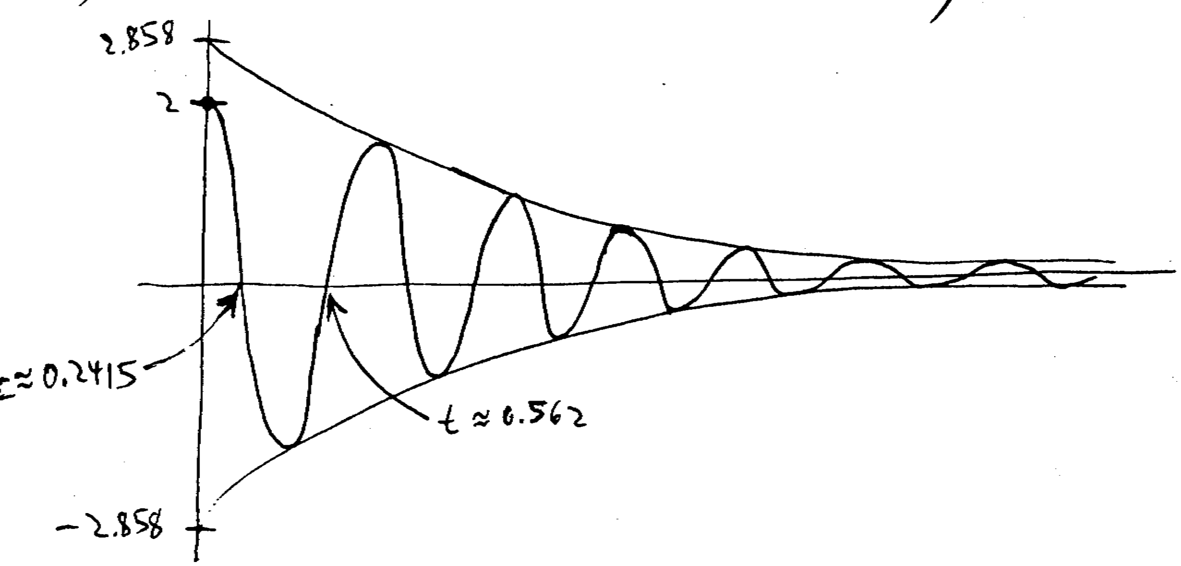
So $u(t) = e^{-10t}\left(2\cos(9.798t) + 2.041\sin(9.798t)\right)$

~~We can give a rough sketch of the solution by observing that since~~

Now to re-express the solution in the form $u(t) = Re^{-10t}\cos(9.798t - \delta)$ we compute $\delta = \tan^{-1}\left(\frac{2.041}{2}\right) \approx 0.7955$. Then on one hand, $R = \frac{2}{\cos\delta} \approx 2.857$; on the other hand, $R = \frac{2.041}{\sin\delta} \approx 2.858$.

Close enough. So $u(t) = 2.858\, e^{-10t}\cos(9.798t - 0.7955)$.

The solution "starts" at the point $(0, 2)$ and $u' = 0$ there, so we should have a horizontal tangent there. Then the solution should initially decrease. First return to equilibrium occurs at the smallest positive value of t for which $\cos(9.798t - 0.7955) = 0$. So that would be when $9.798t - 0.7955 = \frac{\pi}{2} \Rightarrow t \approx 0.2415$ seconds. Then, as we've already observed, the quasi-period is 0.641 seconds, so the load returns to equilibrium (or rather, to the equilibrium position) every $\frac{0.641}{2} = 0.3205$ seconds. Finally, the graph of the solution bounces between the curves $y = 2.858\, e^{-10t}$ and $y = -2.858\, e^{-10t}$. Putting all this together, we get the following sketch:



Finally, we are asked (essentially) to compare the quasiperiod here to the period of motion we would observe, with the same spring and the same load, if damping were removed. So this would give rise to the diff. eq. $u'' + 196u = 0$, solutions to which are of the form

$$u(t) = A\cos(\underbrace{\sqrt{196}}_{14}\, t) + B\sin(14t) = R\cos(14t - \delta).$$

So the frequency would be 14; the motion would have period $\frac{2\pi}{14}$ ≈ 0.449 seconds.

So the requested ratio is $\frac{0.641}{0.449} \approx 1.43$, meaning that the fluid resistance is strong enough to increase the time intervals between successive returns to equilibrium by about 43% over undamped motion.

(13) If $m = k = 1$, then what we called "small damping" in lecture occurs for $0 < \gamma < 2$. (otherwise we have no oscillation and so there is no quasiperiod to discuss.) And in this case solutions take the form $u(t) = e^{-\frac{\gamma}{2}t}(A\cos(\beta t) + B\sin(\beta t))$ with $\beta = \frac{\sqrt{4-\gamma^2}}{2}$.

So the quasiperiod would be $\dfrac{2\pi}{\frac{\sqrt{4-\gamma^2}}{2}} = \dfrac{4\pi}{\sqrt{4-\gamma^2}}$

The corresponding undamped motion would obey the diff. eq. $u'' + u = 0$, solutions to which are $A\cos t + B\sin t$ with period 2π. So we want $\dfrac{4\pi}{\sqrt{4-\gamma^2}} = \left(\begin{matrix}50\% \text{ greater} \\ \text{than } 2\pi\end{matrix}\right) = 1.5\,(2\pi) = 3\pi$

$\Rightarrow \sqrt{4-\gamma^2} = \frac{4}{3} \Rightarrow \gamma = \sqrt{\frac{20}{9}} \approx 1.491.$

So $u'' + 1.491\,u' + u = 0$ is the governing equation for the damped oscillation so described.

(17) $w = mg = KL \Rightarrow 8 \text{ lb.} = K\,(1.5 \text{ in}) \Rightarrow K = \frac{16}{3}\,\frac{\text{lb}}{\text{in}}$, or $K = 64 \text{ lb./ft.}$ Then $m = \dfrac{w}{g} = \dfrac{8 \text{ lb.}}{32 \text{ ft/sec}^2} = \dfrac{1}{4}\,\dfrac{\text{lb-sec}^2}{\text{ft.}}$. So

$\frac{1}{4}u'' + \gamma u' + 64u = 0$ is the diff. eq., and critical damping occurs when γ is large enough to equal $\underbrace{2\sqrt{mK}} = 2\sqrt{\frac{1}{4}(64)}$

$= 2\sqrt{16} = 8\ \dfrac{\text{lb.-sec.}}{\text{ft.}}$

(By the way, it isn't necessary to remember this if you just remember that critical damping occurs when the characteristic equation no longer has complex roots but instead has one real root. This occurs when $\gamma^2 - 4mK = 0$, because of the quadratic formula applied to $mr^2 + \gamma r + K = 0$.)

Section 3.9

(5) $w = mg = kL \Rightarrow 4\text{ lb.} = k(1.5\text{ in}) \Rightarrow k = 6\ \text{lb}/\text{in}$

Then $m = \dfrac{w}{g} = \dfrac{4\text{ lb}}{32\text{ ft}/\text{sec}^2} = \dfrac{4\text{ lb}}{384\text{ in}/\text{sec}^2} = \dfrac{1}{96}\ \dfrac{\text{lb-sec}^2}{\text{in}}$

So the diff. eq. would be $mu'' + ku = \text{external force} \Rightarrow$

$\frac{1}{96}u'' + 6u = 2\cos 3t$, and the IVP is

$$\left\{\begin{array}{c} \frac{1}{96}u'' + 6u = 2\cos 3t \\ u(0) = 2 \\ u'(0) = 0 \end{array}\right\}$$

(9) $w = mg \Rightarrow m = \dfrac{w}{g} = \dfrac{6\text{ lb}}{384\text{ in}/\text{sec}^2} = \dfrac{1}{64}\ \dfrac{\text{lb-sec}^2}{\text{in}}$, and we

are given $k = 1\text{ lb}/\text{in}$. So the IVP to solve is

$$\left\{\begin{array}{c} \frac{1}{64}u'' + u = 4\cos 7t \\ u(0) = 0 \\ u'(0) = 0 \end{array}\right\}$$

$\leftarrow$ DISPLACEMENT IS MEASURED IN INCHES HERE

So first we note that the general solution to the homogeneous diff. eq. $\frac{1}{64}u'' + u = 0$ is $u(t) = A\cos 8t + B\sin 8t$. Then we seek a particular solution u_p to the diff. eq. $\frac{1}{64}u'' + u = 4\cos 7t$.

So assume the solution u_p is of the form $u_p = A\cos 7t + B\sin 7t$.

(Actually $u_p = A\cos 7t$ will work in this particular case.)

Then $u_p' = -7A\sin 7t + 7B\cos 7t$

$$u_p'' = -49A\cos 7t - 49B\sin 7t.$$

So $\frac{1}{64}u_p'' + u_p = \left(-\frac{49A}{64} + A\right)\cos 7t + \left(-\frac{49B}{64} + B\right)\sin 7t$, and

for this to yield $4\cos 7t$ we need $-\frac{49B}{64} + B = \frac{15}{64}B = 0 \Rightarrow B = 0$

and we need $-\frac{49}{64}A + A = \frac{15}{64}A = 4 \Rightarrow A = \frac{256}{15}$ (≈ 17.067).

So $u_p(t) = \frac{256}{15}\cos 7t$, and the general solution to our diff. eq.

is $u(t) = \frac{256}{15}\cos 7t + A\cos 8t + B\sin 8t$.

Applying the initial condition $u(0) = 0$, we have

$$u(0) = \frac{256}{15}\cos 0 + A\cos 0 = 0 \Rightarrow \frac{256}{15} + A = 0$$

$$\Rightarrow A = -\frac{256}{15}.$$

So $u(t) = \frac{256}{15}(\cos 7t - \cos 8t) + B\sin 8t$,

and $u'(t) = \frac{256}{15}(-7\sin 7t + 8\sin 8t) + 8B\cos 8t$, so applying

the I.C. $u'(0) = 0$ we find $8B\cos 0 = 0 \Rightarrow B = 0$. So the

displacement is given by $u(t) = \frac{256}{15}(\cos 7t - \cos 8t)$.

One way to produce a sketch of the solution is to make use of trig identities to express **u** as a product of trig functions instead of a difference of trig functions. Note that for any α and β,

$$\begin{aligned}\cos(\alpha-\beta) - \cos(\alpha+\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\ &\quad - (\cos\alpha\cos\beta - \sin\alpha\sin\beta) \\ &= 2\sin\alpha\sin\beta\end{aligned}$$

Taking $\alpha = \frac{15t}{2}$ and $\beta = \frac{t}{2}$, we have

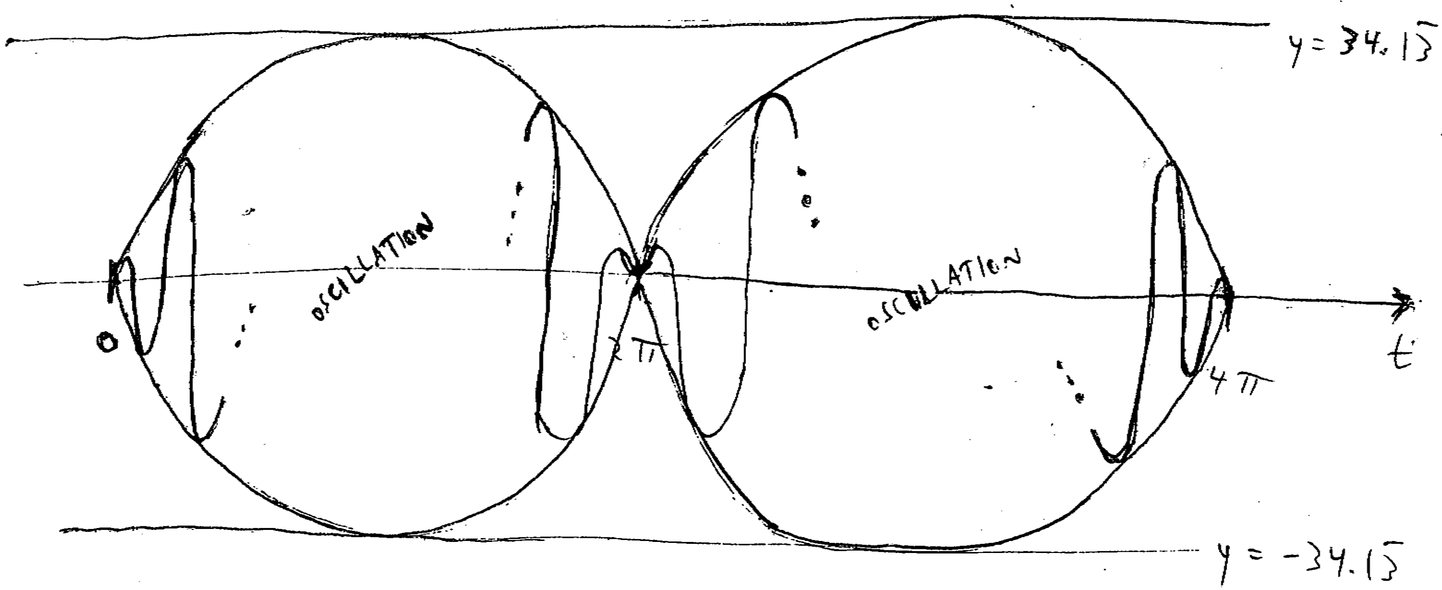
$$u(t) = \frac{256}{15}\cdot 2\cdot \sin\left(\frac{15t}{2}\right)\sin\left(\frac{t}{2}\right) = \underbrace{\frac{512}{15}}_{34.1\overline{3}}\sin(7.5t)\sin(0.5t)$$

From this we can see that whenever both sine functions equal 1, or both equal -1, **u** is equal to $34.1\overline{3}$; when one of the sine functions equals 1 and the other equals -1, u equals $-34.1\overline{3}$. So the maximum displacement from equilibrium is $34.1\overline{3}$ inches, periodically occurring, above and below equilibrium.

Now the first sine function above goes through 7.5 cycles per 2π seconds, while the second sine function completes a half-cycle. So over 4π seconds, one factor completes 14 cycles to the other factor's single cycle. Therefore u is periodic, with period $4\pi \approx 12.57$ seconds.

Note also that $u = 0$ every time $\sin(7.5t) = 0$, so u will hit 0 twenty-seven times in the interval $(0, 4\pi)$ ... and of course, $u(0)$ and $u(4\pi)$ equal 0 as well.

So the upshot is that u will look something like the graph in Figure 3.9.6 on p. 213. In our case, u will be "framed" by the graphs of $y = 34.1\overline{5} \sin(0.5t)$ and $y = -34.1\overline{5} \sin(0.5t)$, and if you don't mind I'd rather not draw 14 cycles, but we'll get a picture something like this:



(11) (a) Here, $k = \dfrac{w}{L} = \dfrac{8 \text{ lb}}{6 \text{ in}}$, and ... OK, for this one I guess I'll break down and use feet instead of inches ... even though I think that's stoooo-pid. <sticking out tongue>

So $k = \dfrac{8 \text{ lb}}{\frac{1}{2} \text{ ft.}} = 16 \dfrac{\text{lb}}{\text{ft}}$. Then we are given $\gamma = \dfrac{1}{4} \dfrac{\text{lb-sec}}{\text{ft}}$.

and $m = \frac{w}{g} = \frac{8 \text{ lb}}{32 \text{ ft/sec}^2} = \frac{1}{4} \frac{\text{lb-sec}^2}{\text{ft}}$.

So the diff. eq. is $\frac{1}{4}u'' + \frac{1}{4}u' + 16u = 4\cos 2t$, with

general solution of this form:

$$u(t) = u_p(t) + \underline{\{\text{general sol'n to homogeneous equation}\}},$$

where u_p is a particular solution to the

equation above. Then, since this part is guaranteed to tend to 0

as $t \to \infty$ (take a gander at Exercise 3.5.38), the steady state is

simply u_p.

So assuming $u_p = A\cos 2t + B\sin 2t$,

$$u_p' = -2A\sin 2t + 2B\cos 2t,$$

and $u_p'' = -4A\cos 2t - 4B\sin 2t$

we want $\frac{1}{4}u_p'' + \frac{1}{4}u_p' + 16u_p = -A\cos 2t - B\sin 2t$

$$-\frac{A}{2}\sin 2t + \frac{B}{2}\cos 2t + 16A\cos 2t + 16B\sin 2t$$

$$= \left(15A + \frac{B}{2}\right)\cos 2t + \left(15B - \frac{A}{2}\right)\sin 2t = 4\cos 2t$$

$$\Longrightarrow \quad 15A + \frac{B}{2} = 4 \quad \text{and} \quad 15B - \frac{A}{2} = 0,$$

whence we get $A = \frac{240}{901}$, $B = \frac{8}{901}$. So the steady state

is $\frac{240}{901}\cos 2t + \frac{8}{901}\sin 2t$ feet.

17 (a) As in #11 (a), the steady state is the solution u_p to the diff. eq. $u'' + \frac{1}{4}u' + 2u = 2\cos\omega t$.

Putting $u_p = A\cos\omega t + B\sin\omega t$, we have

$$u_p' = -\omega A\sin\omega t + \omega B\cos\omega t \quad \text{and}$$

$$u_p'' = -\omega^2 A\cos\omega t - \omega^2 B\sin\omega t, \quad \text{so}$$

$$u_p'' + \frac{1}{4}u_p' + 2u_p = \left(-\omega^2 A + \frac{1}{4}\omega B + 2A\right)\cos\omega t$$

$$+ \left(-\omega^2 B - \frac{1}{4}\omega A + 2B\right)\sin\omega t = 2\cos\omega t$$

$$\Longrightarrow \quad (2-\omega^2)A + \frac{\omega}{4}B = 2$$

$$\text{and} \quad -\frac{\omega}{4}A + (2-\omega^2)B = 0$$

Now since the cosine is an even function, we may as well assume $\omega > 0$. If $\omega = \sqrt{2}$, then we get $A = 0$ and $B = 2\sqrt{2}$, so that $u_p(t) = 2\sqrt{2}\sin(\sqrt{2}t)$, and the amplitude is $2\sqrt{2}$ units

For $\omega \neq \sqrt{2}$, I get $A = \dfrac{32(2-\omega^2)}{(2-\omega^2)^2 + \omega^2}$, $B = \dfrac{8\omega}{(2-\omega^2)^2 + \omega^2}$

(which differs from the back of the book, so it may be in error, but I can't find my error).

(b) Now if u_p is re-expressed as $R\cos(\omega t - \delta)$, then $R = \sqrt{A^2 + B^2}$, and this would be the amplitude of the steady state.

Let's set $d = (2-\omega^2)^2 + \omega^2$, and then

$$A^2 + B^2 = \frac{1024(2-\omega^2)^2}{d^2} + \frac{64\omega^2}{d^2}, \quad \text{so}$$

$$\text{amplitude}^{*} = R = \frac{1}{d}\sqrt{1024(2-\omega^2)^2 + 64\omega^2},$$

which is some crazy function of ω that of course you would not want to have to differentiate to find its maximum value. Even if you did compute $R'(\omega)$ and set it equal to 0, good luck finding roots of $R'(\omega) = 0$. Hence the "need" for a computer for parts (c) and (d).

It's an interesting problem. By doing part (d) you can determine a kind of "worst case scenario", a maximum amplitude for the forced response.

* what is called A in the problem