21-260 Spring 2008
Homework #5 Solutions

Section 3.3

① Neither function is a constant multiple of the other $\Rightarrow$ lin. indep. (on any interval)

② we can rewrite g like so: $g(\theta) = \cos 2\theta + 2(1 - \cos^2\theta)$

$= \cos 2\theta - 2\cos^2\theta + 2$. So $g(\theta) = f(\theta) + 2$, which precludes either function being a constant multiple of the other $\Rightarrow$ lin. indep. (on any interval).

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③ It is not completely obvious that neither function is a multiple of the other, so let's look at $W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g$

$$= e^{\lambda t}\cdot \cos(\mu t)\left[\mu e^{\lambda t}\cos(\mu t) + \lambda e^{\lambda t}\sin(\mu t)\right]$$

$$- \left[-\mu e^{\lambda t}\sin(\mu t) + \lambda e^{\lambda t}\cos(\mu t)\right]\cdot e^{\lambda t}\sin(\mu t)$$

$= \mu e^{2\lambda t}\cos^2(\mu t) + \mu e^{2\lambda t}\sin^2(\mu t) = \mu e^{2\lambda t}$. Since $\mu \neq 0$, W is never 0, and so by Thm. 3.3.1, f and g are linearly indep. (on any interval)

④ Since $e^{3(x-1)} = e^{3x-3} = e^{3x}\cdot e^{-3} = \left(\frac{1}{e^3}\right)e^{3x}$, we see that $g(x) = \text{constant}\cdot f(x)$. So this pair of functions is linearly dependent (on any interval).

⑥ We can consider $(-\infty, 0)$ or $(0, \infty)$, and on either interval we see that neither function is a multiple of the other $\Rightarrow$ lin. indep.

⑨ The two functions are linearly indep. on any interval, because if we let $I = (a,b)$, then the function $t\sin^2 t$ is certainly not 0 for all t in I (though it may be 0 for <u>some</u> t). So by Thm. 3.3.1, the functions are lin. indep. on I.

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(10) By the reasoning in #9, the functions are lin. indep. on any interval.

(15) writing $y'' - \frac{t(t+2)}{t^2} y' + \frac{t+2}{t^2} y = 0$, we have

$$p(t) = \frac{-t(t+2)}{t^2} = \frac{-t^2-2t}{t^2} = -1 - \frac{2}{t}.$$

So $-\int p(t)\,dt = \int 1 + \frac{2}{t}\,dt = t + 2\ln|t| + C$,

and $W(t) = c\cdot \exp(t + 2\ln|t|) = c\cdot\exp(t + \ln|t|^2)$

$$= c\cdot\exp(t+\ln t^2) = c\cdot e^t \cdot e^{\ln t^2} = c t^2 e^t.$$

So if we were to find a pair y_1 and y_2 of solutions (on either $(-\infty,0)$ or $(0,\infty)$) for which $W(y_1,y_2) = ct^2e^t$ with c not zero, we would have a lin. indep. pair (and therefore a fundamental set), and the general solution to the diff. eq. would then be $y(t) = Ay_1(t) + By_2(t)$.

(16) Write $y'' + \frac{\sin t}{\cos t} y' - \frac{t}{\cos t} y = 0$, or

$y'' + (\tan t)y' - (t\sec t)y = 0$, and now we would look for solutions on intervals which avoid the discontinuities of $\tan t$ and $\sec t$ (which occur at the same values of t, namely, the zeros of the cosine function) ... that is, if we were going to actually look for solutions to this equation, which we're not.

Anyway, as $p(t) = \tan t$, the Wronskian of any two solutions is of the form $c \cdot \exp\left(-\int \tan t \, dt\right)$, and

$$\int \tan t \, dt = \int \frac{\sin t}{\cos t} \, dt = -\ln|\cos t| + C = \ln|\cos t|^{-1} + C$$

$$\left\{ \begin{array}{l} \text{Let } u = \cos t \\ du = -\sin t \, dt \end{array} \right. \qquad = \ln|\sec t| + C$$

So for any solutions y_1, y_2, we have $W(y_1, y_2)$

$$= c\, e^{-\ln|\sec t|} = c\, e^{\ln|\sec t|^{-1}} = c\, e^{\ln|\cos t|} = c|\cos t|.$$

(Now since c can be any number, we can just as well express the general Wronskian, associated with solutions to this diff. eq., as $c \cdot \cos t$ ... since, for instance, if $\cos t < 0$ on the interval of interest (such as on $\frac{\pi}{2} < t < \frac{3\pi}{2}$), and y_1 and y_2 are such that $W(y_1, y_2) = -5|\cos t|$, then this is the same as $5\cos t$.)

(24) In view of the four equivalent statements on p. 157 (middle), if t_0 is the point in question, consider $W(y_1, y_2)(t_0) =$

$$\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = 0 - 0 = 0.$$ So the Wronskian

is 0 at some t_0, so this means that Statement 4 is not true.

Therefore, Statement 1 is not true, and so y_1 and y_2 are not a fundamental set of solutions.

(25) The argument here is the same as in #24, but here we have $W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ 0 & 0 \end{vmatrix}$.

Section 3.4

(21) The characteristic equation is $r^2 + r + 1.25 = 0$ (which can be multiplied by 4 if one desires, to produce $4r^2 + 4r + 5 = 0$).

So the roots are $$r = \frac{-1 \pm \sqrt{1 - 4(1.25)}}{2} = \frac{-1 \pm \sqrt{-4}}{2}$$

$$= \frac{-1 \pm 2i}{2}, \text{ or } r_1 = -\tfrac{1}{2} + i \text{ and } r_2 = -\tfrac{1}{2} - i.$$

So the general solution to the diff. eq. is

$$y(t) = e^{-\frac{t}{2}}(A \cos t + B \sin t)$$

Now applying the initial condition $y(0) = 3$, we have

$$y(0) = e^0 (A \cos 0 + B \sin 0) = \underline{\underline{A = 3}}$$

So now we know y takes the form

$$y(t) = e^{-\frac{t}{2}}(3 \cos t + B \sin t)$$

Then $y'(t) = e^{-\frac{t}{2}}(-3 \sin t + B \cos t) - \frac{1}{2} e^{-\frac{t}{2}}(3 \cos t + B \sin t)$.

So $y'(0) = 1$ gives $e^0 \cdot B - \frac{1}{2} e^0 \cdot 3 = B - \frac{3}{2} = 1 \Rightarrow \underline{\underline{B = \frac{5}{2}}}$

So the solution to the IVP is $y(t) = e^{-\frac{t}{2}}(3 \cos t + \frac{5}{2} \sin t)$.

(38) With $x = \ln t$, we have, by the Chain Rule,

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{1}{t} \cdot \frac{dy}{dx} \qquad (1)$$

Then $\displaystyle \frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(\frac{1}{t} \cdot \frac{dy}{dx}\right) = \frac{1}{t} \cdot \frac{d^2y}{dx^2} \cdot \frac{dx}{dt} - \frac{1}{t^2}\frac{dy}{dx}$

$$= \frac{1}{t} \cdot \frac{d^2y}{dx^2} \cdot \frac{1}{t} - \frac{1}{t^2}\frac{dy}{dx}, \text{ so } \frac{d^2y}{dt^2} = \frac{1}{t^2}\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right) \qquad (2)$$

So then, using (1) and (2), for $t > 0$,

$$t^2 \frac{d^2y}{dt^2} + \alpha t \frac{dy}{dt} + \beta y = \frac{t^2}{t^2}\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right) + \frac{\alpha t}{t}\frac{dy}{dx} + \beta y$$

$$= \frac{d^2y}{dx^2} + (\alpha - 1)\frac{dy}{dx} + \beta y.$$

So we arrive at the constant-coefficient diff. eq.

$$\frac{d^2y}{dx^2} + (\alpha - 1)\frac{dy}{dx} + \beta y = 0.$$

Once this equation is solved, one arrives at the general solution, which is a family of functions of x, and then one simply substitutes $\ln t$ for x to get a family of functions of t, which family is the general solution to the original diff. eq. Cool, huh? Way to go, Euler. Such are the researches that occupied him in his copious spare time.

(42) Here is an Euler equation with $\alpha = -4$ and $\beta = -6$. As we showed in #38, converting to independent variable $x = \ln t$ takes us to

the diff. eq. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 6y = 0$, with characteristic

equation $r^2 - 5r - 6 = 0 \Rightarrow$ $(r-6)(r+1) = 0$.

So the roots are $r_1 = 6$ and $r_2 = -1$. Hence the general solution is

$$y(x) = Ae^{6x} + Be^{-x}.$$

So with $x = \ln t$ we have $y(t) = Ae^{6\ln t} + Be^{-\ln t}$

$$= Ae^{\ln t^6} + Be^{\ln t^{-1}} = At^6 + \frac{B}{t}.$$

So the general solution to the given diff. eq. is

$$(*) \quad y(t) = At^6 + \frac{B}{t} \quad (t > 0).$$

As a partial check on the work, you can verify that t^6 and $\frac{1}{t}$ are particular solutions to the given Euler equation. And if you verify that this pair of solutions has a nonzero Wronskian, then that actually provides a <u>full</u> check on the work, for if that is true, then t^6 and $\frac{1}{t}$ are a fundamental pair of solutions, and so $(*)$ must indeed yield the general solution.

$\boxed{\text{Section 3.5}}$

(14) $y'' + 4y' + 4y = 0$ has characteristic equation $r^2 + 4r + 4 = 0$,

or $(r+2)^2 = 0$, with roots $r_1 = -2$ and $r_2 = -2$. So the general solution is $y(t) = (At + B)e^{-2t}$.

Now $y(-1) = 2$ yields $(-A + B)\cdot e^2 = 2$, or $\boxed{B - A = \frac{2}{e^2}}$,

and since $y'(t) = Ae^{-2t} - 2(At + B)e^{-2t}$, the condition

$y'(-1) = 1$ gives $Ae^2 - 2(-A + B)e^2 = 1$, or $(3A - 2B)e^2 = 1$,

or $\boxed{3A - 2B = \frac{1}{e^2}}$.

From the first boxed equation we have $B = A + \frac{2}{e^2}$, so substituting here gives $3A - 2A - \frac{4}{e^2} = \frac{1}{e^2} \Rightarrow \underline{A = \frac{5}{e^2}}$. So then $\underline{B = \frac{7}{e^2}}$,

and the solution to the IVP is thus $y(t) = \left(\frac{5t}{e^2} + \frac{7}{e^2}\right)e^{-2t}$, or

$y(t) = (5t + 7)e^{-2}\cdot e^{-2t}$, or $y(t) = (5t+7)e^{-2(t+1)}$.

(Any of these three forms is acceptable, as far as I'm concerned.)

2nd part of #14 on top of "page A"

(42) This Euler equation has $\alpha = 2$, $\beta = \frac{1}{4}$, so $x = \ln t$ allows one to convert to the diff. eq.

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{1}{4}y = 0, \qquad (*)$$

with characteristic equation $r^2 + r + \frac{1}{4} = 0 \Leftrightarrow (r + \frac{1}{2})^2 = 0$.

So there is but one real root $r = -\frac{1}{2}$, and the general solution to $(*)$ takes the form ~~$y(t) = (At+B)$~~ (whoops)

$$y(x) = (Ax + B)e^{-\frac{x}{2}}$$

Now substituting $\ln t$ for x gives $y(t) = (A\ln t + B)\,e^{-\frac{1}{2}\ln t}$

$= (A\ln t + B)\,e^{\ln t^{-\frac{1}{2}}} = (A\ln t + B)\cdot\frac{1}{\sqrt{t}}$, so

$$y(t) = A\cdot\frac{\ln t}{\sqrt{t}} + B\cdot\frac{1}{\sqrt{t}} \qquad (\star)$$

describes the general solution to the Euler equation.

Now let's fully check this result, applying what we know from section 3.3. Let $y_1(t) = \frac{\ln t}{\sqrt{t}}$, and observe $y_1'(t) = t^{-\frac{3}{2}}(1 - \frac{1}{2}\ln t)$ and $y_1'' = t^{-\frac{5}{2}}(\frac{3}{4}\ln t - 2)$.

So $t^2 y_1'' + 2t y_1' + \frac{1}{4}y_1 = 0 \Rightarrow y_1$ is indeed a solution to the given eq.

Next let $y_2(t) = \frac{1}{\sqrt{t}}$; then $y_2' = -\frac{1}{2}t^{-\frac{3}{2}}$ and $y_2'' = \frac{3}{4}t^{-\frac{5}{2}}$,

so $t^2 y_2'' + 2t y_2' + \frac{1}{4}y_2 = 0 \Rightarrow y_2$ is a solution as well.

Now check $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = -\frac{1}{2}t^{-\frac{5}{2}}\ln t - t^{-\frac{5}{2}}(1 - \frac{1}{2}\ln t)$

$= -t^{-\frac{5}{2}}$, or $-\frac{1}{\sqrt{t^5}}$, which is never zero on $(0, \infty)$, the interval of interest. So $(\star)$ is indeed the general solution of the given diff. eq.

(14) (continued) Sorry ... forgot the second part. "page A"

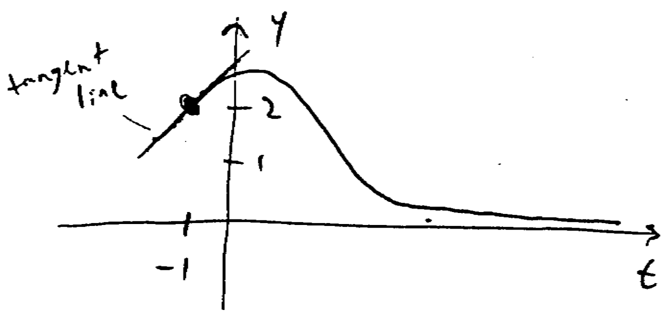
For a rough sketch of the solution, we note that the initial conditions $y(-1) = 2$ and $y'(-1) = 1$ mean that the solution passes through the point $(-1, 2)$ in such a way that the tangent line has slope 1. So the solution is increasing initially. But as $t \to \infty$, we have $y(t)$ tending to 0, because as one can show using l'Hôpital's Rule,

$$\lim_{t \to \infty} (At + B)\, e^{-\alpha t} = 0$$

for all A, B and for any $\alpha > 0$.

Also, in this case, $y(t)$ remains positive for all $t \geq -1$, because $5t + 7 > 0$ for such t, and of course $e^{\text{ANYTHING}} > 0$.

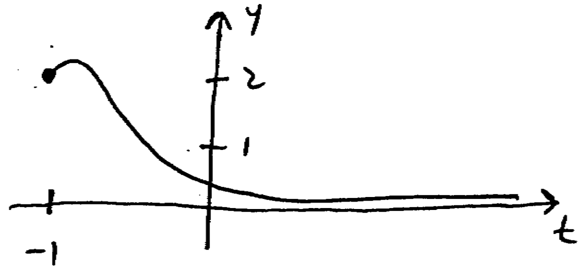
So here's a rough sketch:



Now, though there has to be a maximum point, I don't know that there is only one, unless I verify that. But I find

$$y'(t) = (-10t - 9)\, e^{-2(t+1)},$$

which is 0 only when $t = -\frac{9}{10}$. So in fact there is only one extreme point; also the max. occurs pretty close to the point $(-1, 2)$. So the picture actually looks more like this:



Section 3.6

(7) The homogeneous equation $2y'' + 3y' + y = 0$ has characteristic equation $2r^2 + 3r + 1 = 0 \iff (2r+1)(r+1) = 0$, so the roots are $r_1 = -\frac{1}{2}$ and $r_2 = -1 \Rightarrow$ general solution is

$$y(t) = Ae^{-\frac{t}{2}} + Be^{-t}.$$

Now we look for a particular solution of the given equation taking the form $y_p = At^2 + Bt + C + D\cos t + E\sin t$, which form yields

$$y_p' = 2At + B - D\sin t + E\cos t$$

$$y_p'' = 2A - D\cos t - E\sin t$$

So

$$2y_p'' + 3y_p' + y_p = At^2 + (6A+B)t + (4A + 3B + C)$$
$$+ (3E - D)\cos t + (-E - 3D)\sin t$$

For this function on the right to agree with $t^2 + 3\sin t$, we need

$$\begin{bmatrix} A = 1 \\ 6A + B = 0 \\ 4A + 3B + C = 0 \end{bmatrix} \text{ and } \begin{bmatrix} 3E - D = 0 \\ -E - 3D = 3 \end{bmatrix}$$

This yields $A = 1$, $B = -6$, $C = 14$, $D = -\frac{9}{10}$, $E = -\frac{3}{10}$.

So $y_p = t^2 - 6t + 14 - \frac{9}{10}\cos t - \frac{3}{10}\sin t$

Therefore the general solution to the given diff. eq. consists of all functions of the form

$$y(t) = t^2 - 6t + 14 - \frac{9}{10}\cos t - \frac{3}{10}\sin t + Ae^{-\frac{t}{2}} + Be^{-t}$$

(16) The homogeneous equation $y'' - 2y' - 3y = 0$ has characteristic equation $r^2 - 2r - 3 = 0 \iff (r-3)(r+1) = 0$, with roots $r_1 = 3$ and $r_2 = -1$. So its general solution is $y(t) = Ae^{3t} + Be^{-t}$.

Now we look for a particular solution of the form

$y_p = (At + B)e^{2t} \longrightarrow y_p' = (2At + A + 2B)e^{2t} \Rightarrow$

$y_p'' = (4At + 4A + 4B)e^{2t}$. Then $y_p'' - 2y_p' - 3y_p$

$= (-3At + 2A - 3B)e^{2t}$. So to get $3te^{2t}$ from this, we need

$-3A = 3 \Rightarrow A = -1$, and then $2A - 3B = 0 \Rightarrow -2 - 3B = 0 \Rightarrow$

$B = -\frac{2}{3}$. Hence $y_p(t) = (-t - \frac{2}{3})e^{2t}$, so the general solution to the given diff. eq. is $y(t) = (-t - \frac{2}{3})e^{2t} + Ae^{3t} + Be^{-t}$.

Now, applying the initial condition $y(0) = 1$, we get

$-\frac{2}{3}e^0 + Ae^0 + Be^0 = 1 \Rightarrow \boxed{A + B = \frac{5}{3}}$. Now consider $y'(t)$

$= (-2t - \frac{7}{3})e^{2t} + 3Ae^{3t} - Be^{-t}$, so $y'(0) = 0 \Rightarrow$

$-\frac{7}{3}e^0 + 3Ae^0 - Be^0 = 0 \Rightarrow \boxed{3A - B = \frac{7}{3}}$. From the two boxed equations we get $A = 1$ and $B = \frac{2}{3}$. So the solution to the IVP is $y(t) = (-t - \frac{2}{3})e^{2t} + e^{3t} + \frac{2}{3}e^{-t}$.