21-260 Spring 2008

Homework #4 Solutions

Section 3.1

(15) The characteristic equation, associated with this differential equation, is $r^2 + 8r - 9 = 0$, which factors as $(r+9)(r-1) = 0$.

So the roots are $r_1 = -9$ and $r_2 = 1$; therefore $e^{r_1 t} = e^{-9t}$ and $e^{r_2 t} = e^t$ are solutions to the diff eq., as is every function of the form $y(t) = Ae^{-9t} + Be^t$.

To satisfy the initial conditions we need $y(1) = 1$

$$\Rightarrow \quad Ae^{-9} + Be = 1 \qquad (1)$$

Then $y'(t) = -9Ae^{-9t} + Be^t$, and we need $y'(1) = 0$

$$\Rightarrow \quad -9Ae^{-9} + Be = 0 \qquad (2)$$

From (1) we have $Be = 1 - Ae^{-9}$ and from (2), $Be = 9Ae^{-9}$.

So $1 - Ae^{-9} = 9Ae^{-9} \Rightarrow 10Ae^{-9} = 1 \Rightarrow \boxed{A = \frac{e^9}{10}}$

Then substituting into (2), we have $-\frac{9e^9 \cdot e^{-9}}{10} + Be = 0$

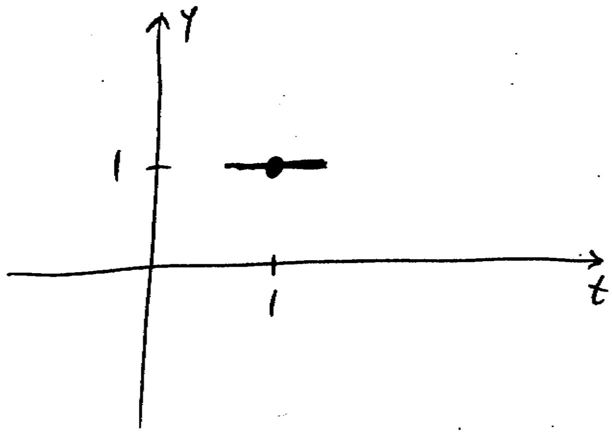
$$\Rightarrow \quad Be = \frac{9}{10} \Rightarrow \boxed{B = \frac{9}{10e}}$$

So the solution to the IVP is $y(t) = \frac{e^9}{10} \cdot e^{-9t} + \frac{9}{10e} \cdot e^t$

Now ... let's go back and consider the initial conditions

$$y(1) = 1$$
$$y'(1) = 0$$

These conditions say that we want a solution passing through (1,1) in such a way that the tangent line is horizontal. That gives this picture:



Now it turned out that A and B were both positive, and given that

$$\lim_{t\to-\infty} Ae^{-9t} = \infty$$

$$\lim_{t\to-\infty} Be^{t} = 0$$

$$\lim_{t\to\infty} Ae^{-9t} = 0$$

$$\lim_{t\to\infty} Be^{t} = \infty$$

for $A, B > 0$, it follows that our solution satisfies

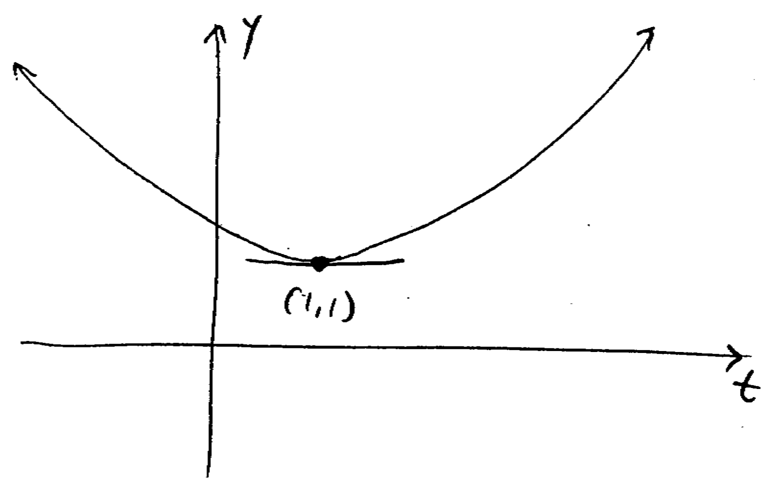
$\lim_{t\to-\infty} y(t) = \lim_{t\to\infty} y(t) = \infty$. Therefore it must have an absolute minimum point. Is (1,1) that point? Or could investigate how many roots $y'(t)$ could have. For $A, B > 0$, $\underbrace{y'(t) = -9Ae^{-9t} + Be^{t} = 0}$ is necessary for a local extremum. Now multiply this by e^{9t}, which preserves the roots because e^{9t} is always positive. So this gives

$$-9A + Be^{10t} = 0 \Rightarrow e^{10t} = \frac{9A}{B}$$

$$\Rightarrow t = \frac{1}{10} \ln\left(\frac{9A}{B}\right)$$

argument here is OK since $A, B > 0$; so $\frac{9A}{B} > 0$

So there is exactly one value of t for which $y'(t) = 0$, which means there can only be one extreme point for y. So for our solution, $(1,1)$ is it! So the solution looks like this:



(17) The differential equation $ay'' + by' + cy = 0$ has such a general solution if the roots of the characteristic equation

$ar^2 + br + c = 0$ are $r_1 = 2$ and $r_2 = -3$. These are the oots of $(r-2)(r-(-3)) = 0$, or $(r-2)(r+3) = 0$, r $r^2 + r - 6 = 0$. So $y'' + y' - 6y = 0$ works (as does, ay, $-37y'' - 37y' + 222y = 0$, or something silly like that, ince the characteristic equation is then $-37r^2 - 37r + 222 = 0$, r $37r^2 + 37r - 222 = 0 \Rightarrow 37(r^2 + r - 6) = 0$).

2) $4y'' - y = 0$ is equivalent to $y'' - \frac{1}{4}y = 0$, or $y'' = \frac{1}{4}y$, nd we have seen that $y_1(t) = e^{\sqrt{1/4}\cdot t}$

~~(a fundamental set of) solutions and that~~

$= e^{\frac{t}{2}}$ and $y_2(t) = e^{-\frac{t}{2}}$ are (a fundamental set of) solutions and that

$$y(t) = Ae^{\frac{t}{2}} + Be^{-\frac{t}{2}}$$

the general solution.

Now the I.C. $y(0) = 2 \Rightarrow Ae^0 + Be^0 = 2$, so $\boxed{A+B=2}$.

hen $y'(t) = \frac{A}{2}e^{\frac{t}{2}} - \frac{B}{2}e^{-\frac{t}{2}}$, and $y'(0) = \beta \Rightarrow$

$\frac{A}{2}e^0 - \frac{B}{2}e^0 = \beta \Rightarrow \boxed{A - B = 2\beta.}$

t this point let's note that $\lim_{t\to\infty} Be^{-\frac{t}{2}} = 0$ regardless of the value of B. So to get $\lim_{t\to\infty} y(t) = 0$ we also need $\lim_{t\to\infty} Ae^{\frac{t}{2}} = 0$.

But this can only happen with $A = 0$. So our boxed equations now reduce to $B = 2$ and $-B = 2\beta \longrightarrow \underline{\underline{\beta = -1}}$.

So of all solutions to this diff. eq. which pass through the point $(0,2)$, only this one has the behavior $\lim_{t\to\infty} y(t) = 0$. All other solutions passing through this point tend either to ∞ or to $-\infty$ as $t \to \infty$.

(24) To get all solutions to tend to 0, we want the general solution to be $y(t) = Ae^{r_1 t} + Be^{r_2 t}$ with r_1 and r_2 both negative. So we want the characteristic equation

$$r^2 + (3-\alpha)r - 2(\alpha - 1) = 0$$

to have two negative roots. The quadratic equation gives the roots

$$r = \frac{\alpha - 3 \pm \sqrt{(3-\alpha)^2 + 4\cdot 2(\alpha-1)}}{2},$$

and the greater of these (if indeed the roots are distinct) is

$$\frac{\alpha-3}{2} + \frac{1}{2}\sqrt{(3-\alpha)^2 + 8(\alpha - 1)}.$$ So if we can get this one to be negative, then the other one is automatically negative as well

OK, so $\frac{\alpha-3}{2} + \frac{1}{2}\sqrt{(3-\alpha)^2 + 8(\alpha-1)} < 0$ is equivalent to

$$\alpha - 3 + \sqrt{(3-\alpha)^2 + 8(\alpha-1)} < 0$$

$$\Rightarrow \sqrt{(3-\alpha)^2 + 8(\alpha-1)} < 3-\alpha$$

$$\Rightarrow \sqrt{\alpha^2 + 2\alpha + 1} < 3 - \alpha$$

$$\Rightarrow \sqrt{(\alpha+1)^2} < 3-\alpha$$

$$\Rightarrow |\alpha + 1| < 3 - \alpha \qquad (*)$$

Now, as this side is nonnegative, we certainly need $3-\alpha > 0$ $\Rightarrow \alpha < 3$. Next let's note that if $\alpha < -1$, then $(*)$ is ivially satisfied, for then $|\alpha+1| = -(\alpha+1) = -\alpha - 1$, hich is certainly less than $-\alpha + 3 = 3 - \alpha$. So all $\alpha < -1$ tisfy $(*)$. So now suppose $\alpha \geq -1$. Then $(*)$ becomes

$$\alpha + 1 < 3 - \alpha \Rightarrow 2\alpha < 2 \Rightarrow \boxed{\alpha < 1}$$

This last condition is more restrictive than our earlier condition $\alpha < 3$. So we need $\alpha < 1$ to get all solutions y satisfying

$$\lim_{t\to\infty} y(t) = 0.$$

For the other part, note that our work above gives the roots the characteristic equation as

$$r = \frac{\alpha - 3 \pm (\alpha + 1)}{2}$$

$$\Longrightarrow \quad r_1 = \frac{\alpha - 3 + \alpha + 1}{2} = \frac{2\alpha - 2}{2} = \alpha - 1$$

and
$$r_2 = \frac{\alpha - 3 - (\alpha + 1)}{2} = \frac{\alpha - 3 - \alpha - 1}{2} = -2$$

(So now it is easier to see that if $\alpha < 1$, then both roots are negative. However, there is one worrisome issue here, which is that if $\alpha = -1$, then both roots are the same, and as of this section – indeed, as of section 3.3 – we do not yet know the form for the general solution to $ay'' + by' + cy = 0$ when the characteristic equation has two equal roots ... or if you prefer, has one "double root." Let's wave that off for now.)

Now then, if $\alpha = 1$, then solutions are of the form

$$y(t) = Ae^{0t} + Be^{-2t} = A + Be^{-2t}$$

Any such solution with $B = 0$ is then just of the form $y \equiv A$, which is to say that all constant functions are solutions. (This makes sense if you look at the resulting diff. eq., which is $y'' + 2y' = 0$.) None of these solutions becomes unbounded, so $\alpha = 1$ is no good. But $\alpha > 1$ is no good, either, because then $r_1 > 0$, and the general solution $y(t) = Ae^{r_1 t} + Be^{-2t}$ includes plenty of bounded solutions*, namely, all those having $A = 0$, which satisfy $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} Be^{-2t} = 0$.

So no value of α works for the second part.

* I should say, bounded as $t \to \infty$

$\boxed{\text{Section 3.2}}$

(11) The initial conditions are imposed at $x_0 = 1$, so we need an interval containing this value of x. Writing the diff. eq. as

$$y'' + \frac{x}{x-3}y' + \frac{\ln|x|}{x-3}y = 0,$$ we can refer to Theorem 3.2.1,

and here, $p(x) = \dfrac{x}{x-3}$ and $q(x) = \dfrac{\ln|x|}{x-3}$. So p has a discontinuity at $x = 3$, and q has discontinuities at $x = 0$ and $x = 3$. So the answer to the question is $(0, 3)$.

(12) As in #11, if we divide through by the y''-coefficient, we have

$$y'' + \frac{1}{x-2}y' + (\tan x)y = 0.$$ So $p(x) = \dfrac{1}{x-2}$, which is

discontinuous at $x = 2$, and $q(x) = \tan x$, which is discontinuous at all odd integer multiples of $\frac{\pi}{2}$. We need an interval containing $x = 3$ and avoiding the discontinuities of q, so that would be $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, but as $\frac{3\pi}{2} \approx 4.71$ and $\frac{\pi}{2} \approx 1.57$, this interval contains $x = 2$, where p is discontinuous. So $(2, \frac{3\pi}{2})$ is the largest interval on which we can be sure that the solution to the IVP is continuous (in fact, twice differentiable) and unique.

(13) If $y(t) = t^2$, then $y'(t) = 2t$, and $y'' \equiv 2$.

So $t^2 y'' - 2y = 2t^2 - 2t^2 \equiv 0$.

And if $y(t) = \frac{1}{t}$, then $y' = -\frac{1}{t^2}$ and $y'' = \frac{2}{t^3}$, so

$t^2 y'' - 2y = \frac{2t^2}{t^3} - \frac{2}{t} \equiv 0$. So yes, y_1 and y_2 are solutions.

The fact that all linear combinations of these two functions are also solutions follows directly from the Superposition Principle, since this diff. eq. is homogeneous. But it's sometimes good to see theorems like 3.2.2 in action in particular cases, so let's just verify the Superposition Principle directly for this equation and this pair of solutions: Let c_1, c_2 be any numbers, and define $y(t) = c_1 t^2 + \frac{c_2}{t}$.

Then $y'(t) = 2c_1 t - \frac{c_2}{t^2}$, and $y''(t) = 2c_1 + \frac{2c_2}{t^3}$.

Then $t^2 y'' - 2y = t^2\left(2c_1 + \frac{2c_2}{t^3}\right) - 2\left(c_1 t^2 + \frac{c_2}{t}\right)$

$= 2c_1 t^2 + \frac{2c_2}{t} - 2c_1 t^2 - \frac{2c_2}{t} \equiv 0$. Yep ... it works.

(16) No, it cannot, because if $y(t) = \sin(t^2)$, then $y'(t) = 2t\cos(t^2)$, and therefore $y(0) = 0$ and $y'(0) = 0$. So if this function were a solution to the diff. eq., then it would have to be the unique solution to the IVP

$$\begin{cases} y'' + p(t)\,y' + q(t)\,y = 0 \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

However, no matter what p and q are (as long as they are continuous on an interval containing $t_0 = 0$), the function $y \equiv 0$ is also a solution to this IVP. So this contradicts uniqueness.

(26) If $y(x) = x$, then $y' \equiv 1$ and $y'' \equiv 0$. So

$$(1 - x\cot x)\,y'' - xy' + y = -x + x \equiv 0.$$

Then if $y(x) = \sin x$, then $y'(x) = \cos x$ and $y''(x) = -\sin x$,

$$(1 - x\cot x)\,y'' - xy' + y = (x\cot x - 1)(\sin x) - x\cos x + \sin x$$

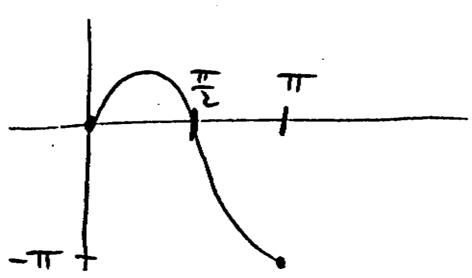
$$= x\cos x - \sin x - x\cos x + \sin x \equiv 0.$$

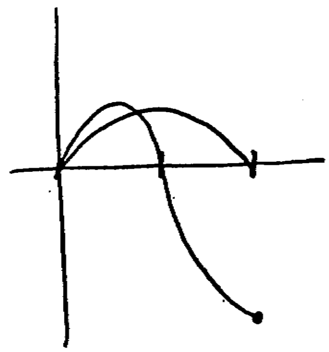
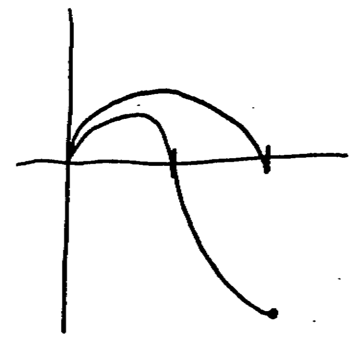
So y_1 and y_2 are solutions. Now we consider $W(y_1, y_2)$

$$= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix} = \underbrace{x\cos x - \sin x}_{\text{call this } W(x)}.$$

Now there are two ways to establish that (on the interval $(0,\pi)$) this pair is a fundamental set of solutions, the easy way and the hard way. But for the easy way, one has to know that it suffices for $W(x_0)$ to be nonzero at any x_0 in $(0,\pi)$. So just take $x_0 = \frac{\pi}{2}$ and note $W(\frac{\pi}{2}) = 0 - \sin\frac{\pi}{2} = -1$. So it is clear that W is not identically zero on $(0,\pi)$; therefore, y_1 and y_2 constitute a fundamental set of solutions, and the general solution (on $(0,\pi)$) is given by $y(x) = Ax + B\sin x$.

The hard way is required if one only knows the definition of fundamental set that asks that W be nonzero for every x in the interval. It's not obvious that $W(x) = x\cos x - \sin x$ is never 0 for $0 < x < \pi$, even if one sketches $f(x) = x\cos x$ and $g(x) = \sin x$ together ~~to see if~~ (on, let's say, $[0,\pi]$) to see if $f(x) = g(x)$ is ever true. To roughly sketch f, note $f(0) = 0$, $f > 0$ on $(0,\frac{\pi}{2})$ since x and $\cos x$ are positive, $f(\frac{\pi}{2}) = 0$, $f < 0$ on $(\frac{\pi}{2},\pi)$, and $f(\pi) = -\pi$. So f looks something like this:



Now if we sketch f and g together, does the picture look like this:  or like this: 

Hopefully the latter. One can "cheat" and use technology to plot the two and see whether indeed $g > f$ on $(0, \pi)$, but another way is to establish that $g'(x) > f'(x)$ for $0 < x < \frac{\pi}{2}$. This will tell us that the lines tangent to g always have greater slopes than those tangent to f, so that it is not possible for the graph of f to get above that of g on the interval $(0, \frac{\pi}{2})$.

Well, $g'(x) = \cos x$ and $f'(x) = \cos x - x \sin x$, and since $\sin x > 0$ on $(0, \frac{\pi}{2})$, indeed, $g'(x) > f'(x)$.

So ... W is never 0 on $(0, \pi)$; therefore y_1 and y_2 do constitute a fundamental set of solutions to the diff. eq.