Mathematical Sciences Department
CARNEGIE MELLON UNIVERSITY

Differential Equations
21-260

# Spring 2008 Exam 3

*No calculator of any kind is permitted. Show all work and give clear explanations.*

NAME: Solutions

| Question | Points | Score | Pres Pt |
|---|---|---|---|
| 1 | 24+1 | | |
| 2 | 24+1 | | |
| 3 | 24+1 | | |
| 4 | 24+1 | | |
| **Total** | 100 | | |

1. (24+1 Points) Draw a solution portrait for the system

$$\begin{aligned} x_1' &= 2x_1 + 3x_2 \\ x_2' &= 3x_1 - 6x_2 \end{aligned}$$

or, if the trajectories are spirals, draw a typical trajectory.

Let $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$. Then $\det(A - rI) = \det\begin{bmatrix} 2-r & 3 \\ 3 & -6-r \end{bmatrix}$

$= (2-r)(-6-r) - 9 = r^2 + 4r - 12 - 9 = r^2 + 4r - 21$

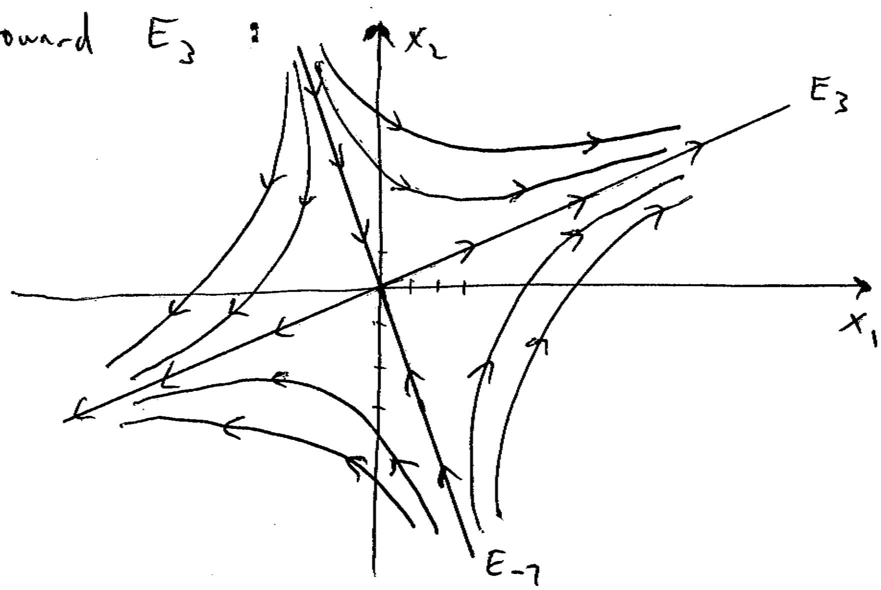
$= (r+7)(r-3) = 0$ if $r = -7$ or $r = 3$.

Find eigenvectors: For $r = -7$, $A - (-7)I = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$, so we

need $\left.\begin{aligned} 9\xi_1 + 3\xi_2 &= 0 \\ 3\xi_1 + \xi_2 &= 0 \end{aligned}\right]$ redundant. Good. $\longrightarrow$ So $\xi_1 = 1$, $\xi_2 = -3$ will work.

For $r = 3$, $A - 3I = \begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} \Rightarrow \left.\begin{aligned} -\xi_1 + 3\xi_2 &= 0 \\ 3\xi_1 - 9\xi_2 &= 0 \end{aligned}\right]$ redundant. me likey. :)

So $\xi_1 = 3$, $\xi_2 = 1$ works. So the eigenspaces are the lines spanned by $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. A trajectory beginning on $E_{-7}$ will stay on $E_{-7}$ and head toward the origin since $r < 0$; a trajectory beginning on $E_3$ will move away from the origin along $E_3$ since $r > 0$. All other trajectories (except the trivial one) tend toward $E_3$:



2. (24+1 Points) Suppose we seek solutions to the differential equation $u_{xx} = u_t + u$ for $0 \le x \le \pi$ and for $t \ge 0$. Suppose the boundary conditions

$$u(0,t) = 0 \text{ and } u(\pi,t) = 0$$

are imposed. Derive the fundamental solutions.

Suppose $u$ is of the form $u(x,t) = X(x)T(t)$. Then $u_{xx} = u_t + u$ becomes $X''T = XT' + XT$. Dividing through by $XT$ gives

$$\frac{X''}{X} = \frac{T' + T}{T} = \frac{T'}{T} + 1.$$ By alternately fixing $x$ and varying $t$, and fixing $t$ while varying $x$, we deduce that $\frac{X''}{X}$ and $\frac{T'}{T} + 1$ must be constant functions and moreover equal to the same constant. Call this constant $-\sigma$. Then we get the ODEs

$$\underline{X'' + \sigma X = 0} \text{ and } T' + (\sigma+1)T = 0.$$

The B.C. $u(0,t) = 0$ for all $t \Rightarrow X(0)T(t) = 0$ for all $t$, which implies $X(0) = 0$ unless $T$ is the zero function (which we don't want). Similarly, $u(\pi,t) = 0$ for all $t \Rightarrow X(\pi) = 0$.

So this together with $X(0) = X(\pi) = 0$ yields a 2-pt BVP; we've seen that nontrivial solutions are only possible if $\sigma > 0$. In that case, let $\lambda^2 = \sigma$ and the gen. sol'n to the ODE is $X(x) = A\cos\lambda x + B\sin\lambda x$. Now $X(0) = 0 \Rightarrow A\cos 0 + B\sin 0 = 0 \Rightarrow A = 0$. Next $X(\pi) = 0 \Rightarrow B\sin\lambda\pi = 0$, which only forces $B = 0$ if $\lambda$ is not an integer. So $X$ can be any multiple of $\sin(nx)$ for any $n = 1, 2, 3, \ldots$

Since this implies $\sigma = n^2$, $T$ must be a solution to $T' + (n^2+1)T = 0 \Rightarrow T(t) = (\text{const.})\exp((-n^2-1)t)$. So the fund. soln's are $\sin(nx)\exp((-n^2-1)t)$ for $n = 1, 2, 3, \ldots$

3. (24+1 Points) Let $f$ be defined over $[0, 2\pi]$ by

$$f(x) = \begin{cases} 0 & \text{for} \quad 0 \le x < \pi/2 \\ 1 & \text{for} \quad \pi/2 \le x \le 2\pi \end{cases}$$

Find a Fourier series for $f$, valid over $[0, 2\pi]$ except possibly for finitely many $x$ values.

We can extend $f$ to $[-2\pi, 2\pi]$ as either an even or an odd function.

I'll extend to an even function by defining

$$\bar{f}(x) = \begin{cases} 0 & \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 1 & \text{otherwise} \end{cases}$$

[Then we'll have $b_n = 0$ for every $n$]

Then $a_0 = \frac{1}{2\pi} \int_{-2\pi}^{2\pi} \bar{f}(x)\, dx = \frac{1}{2\pi}\left( \int_{-2\pi}^{-\frac{\pi}{2}} 1\, dx + \int_{\frac{\pi}{2}}^{2\pi} 1\, dx \right)$

$= \frac{1}{2\pi}\left(\frac{3\pi}{2} + \frac{3\pi}{2}\right) = \frac{3}{2}$

Now for $n > 0$, $a_n = \frac{1}{2\pi} \int_{-2\pi}^{2\pi} \bar{f}(x) \cos\left(\frac{n\pi x}{2\pi}\right) dx$

$= \frac{1}{2\pi} \int_{-2\pi}^{2\pi} \bar{f}(x) \cos\left(\frac{nx}{2}\right) dx = \frac{1}{2\pi}\left( \int_{-2\pi}^{-\frac{\pi}{2}} \cos\left(\frac{nx}{2}\right) dx + \int_{\frac{\pi}{2}}^{2\pi} \cos\left(\frac{nx}{2}\right) dx \right)$

$= \frac{1}{2\pi}\left( \left[\frac{2}{n} \sin\left(\frac{nx}{2}\right) dx\right]_{-2\pi}^{-\frac{\pi}{2}} + \left[\frac{2}{n} \sin\left(\frac{nx}{2}\right)\right]_{\frac{\pi}{2}}^{2\pi} \right)$

$= \frac{1}{2\pi}\left( -\frac{2}{n} \sin\left(\frac{n\pi}{4}\right) - 0 + 0 - \frac{2}{n} \sin\left(\frac{n\pi}{4}\right) \right) = -\frac{2}{n\pi} \sin\left(\frac{n\pi}{4}\right)$

We'll leave $a_n$ like this because the possible values are $\pm \frac{1}{\sqrt{2}}$, $\pm 1$, $0$, and breaking down to cases is a bit cumbersome.

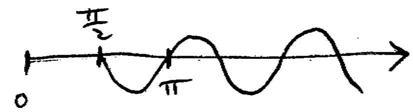
So a F.S. representation (in this case a F.C.S.) for $f$ is

$$\frac{3}{4} + \sum_{n=1}^{\infty} \left[ -\frac{2}{n\pi} \sin\left(\frac{n\pi}{4}\right) \right] \cos\left(\frac{nx}{2}\right)$$

4. (24+1 Points) Solve the initial value problem.

$$\begin{aligned} y'' + 4y &= u_{\pi/2}(t)\cdot\cos t \\ y(0) &= 1 \\ y'(0) &= 0 \end{aligned}$$

graph of this:




(translation of $-\sin t$)

So $u_{\frac{\pi}{2}}(t)\cos t$ can also be written as

$u_{\frac{\pi}{2}}(t)\sin(t-\frac{\pi}{2})$. The Laplace transform is then $e^{-\frac{\pi}{2}s}\mathcal{L}\{\sin t\}$

$= \dfrac{e^{-\frac{\pi}{2}s}}{s^2+1}$ so this equals $\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = s^2\mathcal{L}\{y\} - sy(0)$

$-y'(0) + 4\mathcal{L}\{y\} = (s^2+4)\mathcal{L}\{y\} - s$. Solving for $\mathcal{L}\{y\}$ gives

$\mathcal{L}\{y\} = \dfrac{e^{-\frac{\pi}{2}s}}{(s^2+1)(s^2+4)} + \dfrac{s}{s^2+4} = \mathcal{L}\{??\} + \mathcal{L}\{\cos 2t\}$. Now let's

determine what function $g$ yields $\mathcal{L}\{g\} = \dfrac{1}{(s^2+1)(s^2+4)}$, which we

write as $\dfrac{As+B}{s^2+1} + \dfrac{Cs+D}{s^2+4}$. Then $1 = (As+B)(s^2+4) + (Cs+D)(s^2+1)$

$= (A+C)s^3 + (B+D)s^2 + (4A+C)s + (4B+D)$. So $A+C=0 \Rightarrow C=-A$.

Then $4A+C=0 \Rightarrow 4A-A=0 \Rightarrow 3A=0 \Rightarrow A=0 \Rightarrow C=0$.

Next $B+D=0 \Rightarrow D=-B$, and $4B+D=1 \rightarrow 4B-B=1 \Rightarrow 3B=1$

$\Rightarrow B=\frac{1}{3}$ and $D=-\frac{1}{3}$.

So $\mathcal{L}\{g\} = \dfrac{1/3}{s^2+1} - \dfrac{1/3}{s^2+4} = \mathcal{L}\{\frac{1}{3}\sin t - \frac{1}{6}\sin 2t\}$.

So the solution to the IVP is

$$y(t) = \frac{1}{3}u_{\frac{\pi}{2}}(t)\sin(t-\tfrac{\pi}{2}) - \frac{1}{6}u_{\frac{\pi}{2}}(t)\sin(2t-\pi) + \cos 2t$$