Section 6.1

Exercise 10: Find a unit vector $\mathbf{u}$ in the direction of the given vector

$$\mathbf{w} = \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}.$$

Solution. There are two solutions:

$$\mathbf{u} = \frac{\pm 1}{\|\mathbf{w}\|}\mathbf{w} = \frac{\pm 1}{\sqrt{36+16+9}} \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix} = \pm \begin{bmatrix} -6/\sqrt{61} \\ 4/\sqrt{61} \\ -3/\sqrt{61} \end{bmatrix}.$$

∎

Exercise 24: Verify the *paralellograph law*

$$\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2.$$

Solution. Let $\mathbf{u} = (u_1, \dots, u_n)$ and $\mathbf{v} = (v_1, \dots, v_n)$. We have

$$\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = \sum_{i=1}^{n}(u_i+v_i)^2 + \sum_{i=1}^{n}(u_i-v_i)^2 = \sum_{i=1}^{n}(2u_i^2+2v_i^2) = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2.$$

(Here we used the simple identity $(a+b)^2 + (a-b)^2 = 2a^2 + 2b^2$ valid for any scalars a, b.) ∎

Section 6.2

Exercise 30: Let U be orthonormal matrix, and construct V by interchanging some of the rows of U. Explain why V is orthonormal.

Solution 1. Let us recall some implications proved in the class. By definition, a matrix U is orthogonal if and only if $U^TU = I$. Since both U and U^T are square matrices, the latter identity is equivalent to $UU^T = I$ by the Invertible Matrix Theorem. But $UU^T = I$ is equivalent to the rows of U being orthonormal.

In summary, U is othogonal if and only if its rows are orthonormal. The latter property is clearly preserved by any row permutation.

Solution 2 (or rather a hint). Observe that the dot product $\mathbf{x} \cdot \mathbf{y}$ does not change if the entries of $\mathbf{x}$ are permuted in the same way as the entries of $\mathbf{y}$. ∎

Section 6.3

Exercise 7: Let $W = \text{Span}\{\mathbf{u}_2, \mathbf{u}_2\}$. Write $\mathbf{y}$ as the sum of a vector in W and a vector orthogonal to W. Here

$$\mathbf{y} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}.$$

Solution. The vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal to each other. First we compute $\hat{\mathbf{y}}$, the orthogonal projection of $\mathbf{y}$ onto W:

$$\hat{\mathbf{y}} = \frac{1+9-10}{1+9+4}\,\mathbf{u}_1 + \frac{5+3+20}{25+1+16}\,\mathbf{u}_2 = 0\,\mathbf{u}_1 + \frac{2}{3}\,\mathbf{u}_2 = \begin{bmatrix} 10/3 \\ 2/3 \\ 8/3 \end{bmatrix}.$$

Let $\mathbf{v} = \mathbf{y} - \hat{\mathbf{y}} = (-7/3, 7/3, 7/3)$. Then $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{v}$ is the required sum. ∎

Exercise 8: Let $W = \text{Span}\{\mathbf{u}_2, \mathbf{u}_2\}$. Write $\mathbf{y}$ as the sum of a vector in W and a vector orthogonal to W. Here

$$\mathbf{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}.$$

Solution. The vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal to each other. First we compute $\hat{\mathbf{y}}$, the orthogonal projection of $\mathbf{y}$ onto W:

$$\hat{\mathbf{y}} = \frac{-1+4+3}{1+1+1}\,\mathbf{u}_1 + \frac{1+12-6}{1+9+4}\,\mathbf{u}_2 = 2\,\mathbf{u}_1 + \frac{1}{2}\,\mathbf{u}_2 = \begin{bmatrix} 3/2 \\ 7/2 \\ 1 \end{bmatrix}.$$

Let $\mathbf{v} = \mathbf{y} - \hat{\mathbf{y}} = (-5/2, 1/2, 2)$. Then $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{v}$ is the required sum. ∎

Section 6.4

Exercise 10: Find an orthogonal basis for the column space of

$$A = [\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3] = \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}.$$

Solution. We apply the Gram-Schmidt process. We let $\mathbf{v}_1 = \mathbf{x}_1$. Also,

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{-6-24-2-4}{1+9+1+1}\mathbf{v}_1 = \mathbf{x}_2 + 3\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}.$$

Finally, we should let

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{-6+9+6-3}{1+9+1+1}\mathbf{v}_1 - \frac{18+3+6+3}{9+1+1+1}\mathbf{v}_2 = \mathbf{x}_3 - \frac{1}{2}\mathbf{v}_1 - \frac{5}{2}\mathbf{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}.$$

A routine checking shows that the obtained vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are indeed orthogonal. ∎

Section 6.5

Exercise 12: Find (a) the orthogonal projection of $\mathbf{b}$ into $\operatorname{Col} A$ and (b) a least-square solution of $A\mathbf{x} = \mathbf{b}$. Here

$$A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix}.$$

Solution. It is easy to check that the columns of A are orthogonal to each other. (In particular, they are linearly independent.) Hence, we can use the standard formulas for finding the orthogonal projection of $\mathbf{b}$ onto $\operatorname{Col} A$:

$$\hat{\mathbf{b}} = \frac{2+5-6}{1+1+1}\mathbf{v}_1 + \frac{2+6+6}{1+1+1}\mathbf{v}_2 + \frac{-5+6-6}{1+1+1}\mathbf{v}_1 = \frac{1}{3}(\mathbf{v}_1 + 14\mathbf{v}_2 - 5\mathbf{v}_3) = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix}.$$

This answers (a). Since the columns of A are linearly independent, the least-square solution $\mathbf{x}$ is unique and we already know the weights, namely $\mathbf{x}(= 1/3, 14/3, -5/3)$.

As an check, one can compute $\mathbf{b} - \hat{\mathbf{b}} = (-3, 3, 3, 0)$ and see that it is indeed orthogonal to each $\mathbf{v}_i$. ∎

Section 6.6

Exercise 4: Find the equation $y = \beta_0 + \beta_1 x$ of the least-squares line that fits best the given data points:

$$(2,3),\ (3,2),\ (5,1),\ (6,0).$$

Solution. We construct the design matrix and the observation vector:

$$X = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 5 \\ 1 & 6 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}.$$

We want to find the least-squares solution to $X\bar{\beta} = \mathbf{y}$. The normal equation is

$$X^T X\bar{\beta} = X^T\mathbf{y}.$$

We have

$$X^T X = \begin{bmatrix} 4 & 16 \\ 16 & 74 \end{bmatrix}, \quad X^T\mathbf{y} = \begin{bmatrix} 6 \\ 17 \end{bmatrix}.$$

It is probably easier to compute first the inverse

$$(X^T X)^{-1} = \frac{1}{4 \cdot 74 - 16^2} \begin{bmatrix} 74 & -16 \\ -16 & 4 \end{bmatrix} = \frac{1}{40} \begin{bmatrix} 74 & -16 \\ -16 & 4 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 37 & -8 \\ -8 & 2 \end{bmatrix}.$$

Hence, the least-squares solution is

$$\bar{\beta} = \frac{1}{20} \begin{bmatrix} 37 & -8 \\ -8 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 17 \end{bmatrix} = \begin{bmatrix} 43/10, -7/10 \end{bmatrix}.$$

Thus the least-squares line is $y = 4.3 - 0.7x$. ∎

Exercise 6: Let X be the design matrix corresponding to a least-squares fit of a parabola to data $(x_1, y_1), \dots, (x_n, y_n)$. Suppose that x_1, x_2, x_3 are distinct. Explain why there is only one parabola that fits the data best, in a least-square sence.

Solution. It is enough to prove that the columns of X are linearly independent, since then $X^T X$ is invertible and the unique least-squares solution is $(X^T X)X^T\mathbf{y}$.

Let us remove Row 1 from any other row of X:

$$X = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ & \dots & \end{bmatrix} \sim \begin{bmatrix} 1 & x_1 & x_1^2 \\ 0 & x_2 - x_1 & x_2^2 - x_1^2 \\ 0 & x_3 - x_1 & x_3^2 - x_1^2 \\ & \dots & \end{bmatrix}$$

Calculations show that the determinant

$$\begin{vmatrix} x_2 - x_1 & x_2^2 - x_1^2 \\ x_3 - x_1 & x_3^2 - x_1^2 \end{vmatrix} = (x_2 - x_1)(x_3 - x_1) \begin{vmatrix} 1 & x_2 + x_1 \\ 1 & x_3 + x_1 \end{vmatrix} = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2).$$

This is non-zero since x_1, x_2, x_3 are distinct by the assumption. Thus this 2×2-matrix is invertible and has 2 pivot columns. This means that if we continue the row reduction of X, then we get 3 pivots. Thus the columns of X are independent, as required. ∎

Section 7.1

Exercise 14: Orthogonally diagonalize matrix

$$A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}.$$

Solution. The characteristic equation is $(1-\lambda)^2 - 25 = 0$. The roots are -4 and 6. We have

$$A + 4I = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad A - 6I = \begin{bmatrix} -5 & 5 \\ 5 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

The corresponding eigenvectors are $\mathbf{v}_1 = (-1, 1)$ and $\mathbf{v}_2 = (1, 1)$. They are orthogonal as we expected them to be. Let us normalize them, by multiplying each by $1/\sqrt{2}$. We let

$$P = \frac{1}{\sqrt{2}} [\mathbf{v}_1 \ \mathbf{v}_2] = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \quad D = \begin{bmatrix} -4 & 0 \\ 0 & 6 \end{bmatrix}.$$

Since P is orthogonal, we have $P^{-1} = P^T$.

$$A = PDP^{-1} = PDP^T$$

is the required othogonal diagonalization. ∎

Exercise 22: Orthogonally diagonalize matrix A, given that its eigenvalues are 0 and 2, where

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Solution. Let us find the eigenvectors corresponding to the eigenvalue $\lambda = 0$, which amounts to finding a basis for the $\operatorname{Nul} A$. We have

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We have one free variable x_4, so $\operatorname{Nul} A$ is 1-dimensional and it spanned by $\mathbf{v}_1 = (0, -1, 0, 1)$. Let us immediately normalize $\mathbf{v}_1$ by replacing it with $\mathbf{v}_1 = (0, -1/\sqrt{2}, 0, 1/\sqrt{2})$.

For $\lambda = 2$, we obtain

$$A - 2I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here x_1, x_3, x_4 are free; the general solution to $(A - 2I)\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Luckily for us, the obtained 3 vectors are already orthogonal, so we just normalize them, having

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}.$$

Now we let

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ 0 & 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Since P is orthogonal, we have $P^{-1} = P^T$.

$$A = PDP^{-1} = PDP^T$$

is the required othogonal diagonalization. ∎

Exercise 32: Suppose that $A = PRP^{-1}$, where P is orthogonal and R is upper triangular. Show that if A is symmetric, then R is symmetric and hence is actually a diagonal matrix.

Solution. By the assumptions we have $P^{-1} = P^T$ and $A^T = A$. This means that

$$PRP^T = A = A^T = (PRP^T)^T = (P^T)^T R^T P^T = PR^T P^T.$$

But $P^T = P^{-1}$ are inverses of each other. So if we multiply the obtained identity by P^{-1} on left and by P on right, we obtain $R = R^T$. Thus R is symmetric. Since all entries of T below the main diagonal are zeros, by symmetry all entries above the main diagonals are zeros too. So R is also diagonal. ∎

Section 7.2

Exercise 10: Let $Q(x_1, x_2) = 9x_1^2 - 8x_1x_2 + 3x^2$. Classify the type of Q and make a change of variables $\mathbf{x} = P\mathbf{y}$ that eliminates all cross-product terms.

Solution. The matrix of Q is

$$A = \begin{bmatrix} 9 & -4 \\ -4 & 3 \end{bmatrix}.$$

First, we find the eigenvalues of A. The characteristic polynomial is

$$P_A(\lambda) = (9-\lambda)(3-\lambda) - 16.$$

Its roots are $\lambda_1 = 1$ and $\lambda_2 = 11$. Both are positive so Q is **positive definite**. (Of course, it is also positive semidefinite but of all types of Q we usally mention the one which is most precise.)

Let us compute the corresponding unit eigenvectors. We have

$$A - I = \begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}.$$

We can take a vector $(1,2)$. After normalizing it by $1/\sqrt{5}$, $\mathbf{v}_1 = (1/\sqrt{5}, 2/\sqrt{5})$. Next,

$$A - 11I = \begin{bmatrix} -2 & -4 \\ -4 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}.$$

Here we take $\mathbf{v}_2 = (2/\sqrt{5}, -1/\sqrt{5})$. The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal as they should be (and each of norm 1). Hence, we take

$$P = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 11 \end{bmatrix}.$$

Then the transformation $\mathbf{x} = P\mathbf{y}$ transforms Q into $y_1^2 + 11y_2^2$.

Now, it could be a good idea to check this by hand. We have $x_1 = y_1/\sqrt{5} + 2y_2/\sqrt{5}$ and $x_2 = 2y_1/\sqrt{5} - y_2/\sqrt{5}$. Then

$$\begin{aligned} Q &= 9(y_1/\sqrt{5} + 2y_2/\sqrt{5})^2 - 8(y_1/\sqrt{5} + 2y_2/\sqrt{5})(2y_1/\sqrt{5} - y_2/\sqrt{5}) + 3(2y_1/\sqrt{5} - y_2/\sqrt{5})^2 \\ &= \frac{1}{5}(9(y_1^2 + 4y_1y_2 + 4y_2^2) - 8(2y_1^2 + 3y_1y_2 - 2y_2^2) + 3(4y_1^2 - 4y_1y_2 + y_2^2) \;=\; y_1^2 + 11y_2^2. \end{aligned}$$

So, everything is OK! ▮

Section 7.3

Exercise 6: Let $Q(\mathbf{x}) = 7x_1^2 + 3x_2^2 + 3x_1x_2$. Find a) the maximum of $Q(\mathbf{x})$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$, b) a unit vector where this maximum is attained, and c) the maximum of $Q(\mathbf{x})$ subject to the constraints $\mathbf{x}^T\mathbf{x} = 1$ and $\mathbf{x}^T\mathbf{u} = 0$.

Solution. The matrix of Q is

$$A = \begin{bmatrix} 7 & 3/2 \\ 3/2 & 3 \end{bmatrix}.$$

Its eigenvalues are $\lambda_1 = 5/2$ and $\lambda_2 = 15/2$ with eigenvectors $\mathbf{v}_2 = (-1,3)$ and $\mathbf{v}_1 = (3,1)$. Hence the answer to a) is $15/2$; the answer to b) is $\mathbf{v}_1/\|\mathbf{v}_1\| = (3/\sqrt{10}, 1/\sqrt{10})$; the answer to c) is $5/2$. ▮

Section 7.4

Exercise 10: Find an SVD of

$$A = \begin{bmatrix} 4 & -2 \\ 2 & -1 \\ 0 & 0 \end{bmatrix}.$$

Solution. In **Step 1** we orthogonally diagonalize

$$A^T A = \begin{bmatrix} 20 & -10 \\ -10 & 5 \end{bmatrix}.$$

Its eigenvalues are $\lambda_1 = 25$ and $\lambda_2 = 0$. (We list them in decreasing order.) The corresponding normalized eigenvectors are $\mathbf{v}_1 = (-2/\sqrt{5}, 1/\sqrt{5})$ and $\mathbf{v}_2 = (1/\sqrt{5}, 2/\sqrt{5})$. By the way, although we are not required to do this, we can immediately write an orthogonal diagonalization $A^T A = V D V^T$, where

$$V = [\mathbf{v}_1\ \mathbf{v}_2] = \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}, \quad D = \begin{bmatrix} 25 & 0 \\ 0 & 0 \end{bmatrix}.$$

At this point it is a good idea to check if $A^T A V = V D$.

In **Step 2** we take the same matrix V as above, but the middle matrix Σ should have dimensions 3×2, the same as those of A, so we just add a row of zeros:

$$\Sigma = \begin{bmatrix} 25 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

In **Step 3** we construct U. We see that A (or equivalently Σ) has rank $r = 1$. So the first column of U is

$$\mathbf{u}_1 = \frac{A\mathbf{v}_1}{\|A\mathbf{v}_1\|} = \frac{(-10/\sqrt{5}, -5/\sqrt{5}, 0)}{\sqrt{20+5}} = \begin{bmatrix} -2/\sqrt{5} \\ -1/\sqrt{5} \\ 0 \end{bmatrix}.$$

We choose the remaining columns $\mathbf{u}_2$ and $\mathbf{u}_3$ of U so that U is orthogonal. This is the same as finding an orthonormal basis of

$$\mathrm{Nul}\,(\mathbf{u}_1^T) = \{\mathbf{u} \in \mathbb{R}^3 \mid \mathbf{u}_1^T \mathbf{u} = 0\}.$$

We have

$$\mathbf{u}_1^T \sim \begin{bmatrix} 1 & 1/2 & 0 \end{bmatrix},$$

so a basis for $\mathrm{Nul}\,(\mathbf{u}_1^T)$ is $\mathbf{w}_2 = (1, -2, 0)$ and $\mathbf{w}_3 = (0, 0, 1)$. Luckily for us these vectors are already orthogonal. (If they were not, then we would have to apply the Gram-Schmidt process to them.) So, it remains only to normalize them, obtaining $\mathbf{u}_2 = (1/\sqrt{5}, -2/\sqrt{5}, 0)$ and $\mathbf{u}_3 = (0, 0, 1)$. We take $U = [\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3]$. Thus the required SVD of A is

$$A = \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ -1/\sqrt{5} & -2\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 25 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}.$$

Finally, we are done! $\blacksquare$