Section 2.1

Exercise 6: We have to compute the product AB in two ways, where

$$A = \begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}.$$

Solution 1. Let $\mathbf{b}_1 = (1,2)$ and $\mathbf{b}_2 = (3,-1)$ be the columns of B. Then $A\mathbf{b}_1 = (0,-3,13)$ and $A\mathbf{b}_2 = (14,-9,4)$. Thus

$$AB = [A\mathbf{b}_1 \; A\mathbf{b}_2] = \begin{bmatrix} 0 & 14 \\ -3 & -9 \\ 8 & 4 \end{bmatrix}.$$

Solution 2.

$$AB = \begin{bmatrix} 4 \cdot 1 + (-2) \cdot 2 & 4 \cdot 3 + (-2) \cdot (-1) \\ (-3) \cdot 1 + 0 \cdot 2 & (-3) \cdot 3 + 0 \cdot (-1) \\ 3 \cdot 1 + 5 \cdot 2 & 3 \cdot 3 + 5 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 0 & 14 \\ -3 & -9 \\ 13 & 4 \end{bmatrix}.$$

Both answers are the same, which is reassuring! ∎

Exercise 18: Suppose the first two columns, $\mathbf{b}_1$ and $\mathbf{b}_2$ of B are equal. What can you say about the columns of AB (if AB is defined)?

Solution. If B has m columns $\mathbf{b}_1, \dots, \mathbf{b}_m$, then

$$AB = [A\mathbf{b}_1 \; A\mathbf{b}_2 \; \dots \; A\mathbf{b}_m].$$

We conclude that the first two columns of AB are equal too. ∎

Section 2.2

Exercise 18: Suppose that P is ivertible and $A = PBP^{-1}$. Solve for B in terms of A.

Solution. Multiply both sides of the equation by P^{-1} from left and by P from right to obtain

$$P^{-1}AP = P^{-1}(PBP^{-1})P = (P^{-1}P)B(P^{-1}P) = IBI = B.$$

Thus $B = P^{-1}AP$. (Note that A and $P^{-1}AP$ are in general two different matrices.) ∎

Exercise 24: Suppose A is $n \times n$ and the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b} \in \mathbb{R}^n$. Explain why A must be invertible.

Solution. The assumption means that the columns of A span $\mathbb{R}^n$. This is equivalent to having a pivot in every row (by Theorem 4 in Section 1.4). But the number of rows is equal the number of columns, so each column has a pivot. Thus we have n pivots on the main diagonal, which means that A is row equivalent to I_n. We conclude that A is invertible by Theorem 7 of Section 2.2. ▮

Exercise 30: To find the inverse of A, we row reduce the augmented matrix $[A\ I]$:

$$\begin{bmatrix} 5 & 10 & 1 & 0 \\ 4 & 7 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 5 & 10 & 1 & 0 \\ 0 & -1 & -4/5 & 1 \end{bmatrix} \sim \begin{bmatrix} 5 & 0 & -7 & 10 \\ 0 & 1 & 4/5 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -7/5 & 2 \\ 0 & 1 & 4/5 & -1 \end{bmatrix}$$

Thus A is invertible and $A^{-1} = \begin{bmatrix} -7/5 & 2 \\ 4/5 & -1 \end{bmatrix}$. ▮

Section 2.3

Exercise 36: Let T be a linear transformation that maps $\mathbb{R}^n$ onto $\mathbb{R}^n$. Show that T^{-1} exists and maps $\mathbb{R}^n$ onto $\mathbb{R}^n$. Is T^{-1} also one-to-one?

Solution. Let A be the stardard $n \times n$-matrix for T. By the assumption on T, the columns of A span $\mathbb{R}^n$. Hence, by the Invertible Matrix Theorem A is an invertible matrix. The linear trasformation $S : \mathbb{R}^n \to \mathbb{R}^n$ defined by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the inverse of T.

The linear transformation S is onto because any $\mathbf{x} \in \mathbb{R}^n$ is the image under S of $T(\mathbf{x})$: $\mathbf{x} = S(T(\mathbf{x}))$. It is also one-to-one because if $S(\mathbf{y}) = \mathbf{0}$, then $\mathbf{y} = T(S(\mathbf{y})) = T(\mathbf{0}) = \mathbf{0}$. Alternatively, the last two claims can be proved by applying the Invertible Matrix Theorem to A^{-1}. ▮

Section 2.7

Exercise 4: Write in homogeneous coordinates the matrix of the 2D transformation which first translates by $(-2, 3)$ and then scales the x-coordinate by .8 and the y-coordinate by 1.2.

Solution. These two operations are given by the corresponding matrices:

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} .8 & 0 & 0 \\ 0 & 1.2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence, their composition is given by

$$BA = \begin{bmatrix} 0.8 & 0 & -1.6 \\ 0 & 1.2 & 3.6 \\ 0 & 0 & 1 \end{bmatrix}.$$

∎

Section 2.8

Exercise 30: Suppose the columns of a matrix $A = [\mathbf{a}_1, \ldots, \mathbf{a}_p]$ are linearly independent. Explain why $\{\mathbf{a}_1, \ldots, \mathbf{a}_p\}$ is a basis for Col$A$.

Solution. The pivot columns of A form a basis for Col$A$. Since the columns of A are linearly independent, every column has a pivot, that is, is a pivot column. ∎

Section 2.9

Exercise 10: We are given a matrix A and its echelon form:

$$\begin{bmatrix} 1 & -2 & 9 & 5 & 4 \\ 1 & -1 & 6 & 5 & -3 \\ -2 & 0 & -6 & 1 & -2 \\ 4 & 1 & 9 & 1 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 9 & 5 & 4 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We see that Columns 1, 2, and 4 are pivot columns. Hence Col$A$ has a basis made of vectors $(1, 1, -2, 4)$, $(-2, -1, 0, 1)$, and $(5, 5, 1, 1)$. Its dimension is 3.

To find a basis for Nul$A$, we have to bring A to the reduced echelon form:

$$A \sim \begin{bmatrix} 1 & -2 & 9 & 0 & 14 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The free variables are x_3 and x_5 and we can write the general solution to $A\mathbf{x} = \mathbf{0}$ as

$$\mathbf{x} = x_3 \begin{bmatrix} -3 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 7 \\ 0 \\ 2 \\ 1 \end{bmatrix}.$$

Hence, a basis for Nul$A$ consists of the vectors $(-3, 3, 1, 0, 0)$ and $(0, 7, 0, 2, 1)$. Its dimension is 2. ∎

Section 3.1

Exercise 32: What is the determinant of an elementary scaling $n \times n$-matrix with k on the diagonal?

Solution We know that for a diagonal matrix, the determinant is the product of all diagonal entries. Hence, the answer here is $k \times \cdots \times k = k^n$. ∎

October 20: Xiaohui Luo's office hours will be 6:30-8:00pm on this day.

October 28: Exam #2 (Chapters 2 & 3, and Sections 5.1 & 5.2)

A Complete Proof that $P_A(\lambda)$ is a Polynomial

Let me give here a carefull proof by induction that $\det(A-\lambda I)$ is a polynomial in λ. (I did not complete it on the last lecture.)

First, we prove a slightly different statement:

Theorem 1. Let B be $n\times n$-matrix with entries $b_{ij} = c_{ij}\,\lambda + d_{ij}$ for some constants c_{ij}, d_{ij}. Then $P(\lambda) = \det B$ is a polynomial. Moreover, the degree of P is at most n.

Proof. We use induction on n to prove both claims about P. The statement is true for $n = 1$ as then $P(\lambda) = c_{11}\lambda + d_{11}$.

Suppose it is true for $n-1$. Let us prove it for n. By expanding along the first row, we obtain

$$P(\lambda) = \sum_{i=1}^{n} (-1)^{i+1}(c_{1i}\lambda + d_{1i}) \det B_{1i}.$$

By the induction assumption, each $\det B_{1i}$ is a polynomial of degree at most $n-1$. Hence, $P(\lambda)$ is the sum of polynomials of degree at most n, so it is itself a polynomial of degree at most n. ∎

By Theorem 1 we know that $P_A(\lambda) = \det(A-\lambda I)$ is a polynomial of degree at most n. We cannot immediately conclude that the degree is precisely n because, perhaps, all terms λ^n cancelled each other. So we recourse to induction again.

Theorem 2. For any $n\times n$-matrix A, the polynomial $P_A(\lambda) = \det(A-\lambda I)$ has degree precisely n.

Proof. We use induction on n. For $n = 1$ we have $P_A(\lambda) = a_{11} - \lambda$ and the claim is true. (As usual, a_{ij} is the ij-th entry of A.)

Suppose the claim is true for $n-1$. Let us prove it for n. Let $B = A - \lambda I$. By applying the expansion along the first row, we obtain

$$P_A(\lambda) = (a_{11} - \lambda)\det B_{11} - a_{12}\det B_{12} + a_{13}\det B_{13} - \dots$$

By the induction assumption, $\det B_{11} = P_{A_{11}}(\lambda)$ is a polynomial of degree precisely $n-1$. By Theorem 1, each $\det B_{1i}$ with $i \geq 2$ is a polynomial of degree at most $n-1$. Hence, $P_A(\lambda)$ is the sum of one polynomial of degree n (namely $(a_{11}-\lambda)\det B_{11}$) and some polymonials of degree at most $n-1$. Hence, the degree of $P_A(\lambda)$ is precisely n. ∎

Section 3.2

Exercise 24: Using determinats to decide if the given set of vectors is linearly dependent:

$$\begin{bmatrix} 4 \\ 6 \\ -7 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ 6 \end{bmatrix}.$$

Solution. We make a matrix whose columns are these vectors:

$$A = \begin{bmatrix} 4 & -7 & -3 \\ 6 & 0 & -5 \\ -7 & 2 & 6 \end{bmatrix}.$$

This is 3×3-matrix, that is, a square matrix. By the Invertible Matrix Theorem, the colums of A are linearly independent if and only if A is invertible, which happens if and only if its determinant is non-zero. We have

$$\begin{vmatrix} 4 & -7 & -3 \\ 6 & 0 & -5 \\ -7 & 2 & 6 \end{vmatrix} = (-1) \times (-7) \times \begin{vmatrix} 6 & -5 \\ -7 & 6 \end{vmatrix} + (-1) \times 2 \times \begin{vmatrix} 4 & -3 \\ 6 & -5 \end{vmatrix} = 7 + 4 = 11$$

Hence, the vectors are linearly independent. ∎

Section 5.1

Exercise 14: Find a basis for the eigenspace for

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -3 & 0 \\ 4 & -13 & 1 \end{bmatrix}, \quad \lambda = -2.$$

Solution. We have to find the solution set to $A + 2I = \mathbf{0}$. Apply row transforms:

$$A + 2I = \begin{bmatrix} 3 & 0 & -1 \\ 1 & -1 & 0 \\ 4 & -13 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 3 & 0 & -1 \\ 4 & -13 & 3 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & -1 \\ 0 & -9 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{bmatrix}$$

The general solution can be written as $\mathbf{x} = x_3(1/3, 1/3, 1)$. Hence, the basis of this eigenspace consists of one vector $\mathbf{v}$, namely

$$\mathbf{v} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1 \end{bmatrix}.$$

■

Exercise 26: Show that if A^2 is the zero matrix, then the only eigenvalue of A is 0.

Solution. If λ is an eigenvalue of A with an eigenvector $\mathbf{x}$, then

$$A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda\mathbf{x} = \lambda^2\mathbf{x}.$$

On the other hand, we know that $A^2\mathbf{x} = \mathbf{0}$. Thus $\lambda = 0$. ■

Section 5.2

Exercise 12: Find the characteristic polynomial of

$$A = \begin{bmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Solution. It is

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 0 & 1 \\ -3 & 4-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} -1-\lambda & 0 \\ -3 & 4-\lambda \end{vmatrix} = (2-\lambda)(-1-\lambda)(4-\lambda).$$

Although this is not required, we can immediately tell the eigenvalues: $\lambda_1 = -1$, $\lambda_2 = 2$, $\lambda_3 = 4$ and, with some extra work, compute the corresponding eigenvectors.

$$A - \lambda_1 I = \begin{bmatrix} 0 & 0 & 1 \\ -3 & 5 & 1 \\ 0 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} -3 & 5 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -5/3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $\mathbf{v}_1 = (5/3, 1, 0)$ is a corresponding eigenvector.

$$A - \lambda_2 I = \begin{bmatrix} -3 & 0 & 1 \\ -3 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $\mathbf{v}_2 = (1/3, 0, 1)$ is a corresponding eigenvector. Finally,

$$A - \lambda_3 I = \begin{bmatrix} -5 & 0 & 1 \\ -3 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $\mathbf{v}_3 = (0, 1, 0)$ is a corresponding eigenvector. ■

Section 5.3

Exercise 14: Diagonalize the matrix

$$A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

Solution. First we compute the characteristic polynomial by expanding $A - \lambda I$ along the third row:

$$P_A(\lambda) = (5-\lambda) \begin{vmatrix} 4-\lambda & 0 \\ 2 & 5-\lambda \end{vmatrix} = (5-\lambda)^2(4-\lambda).$$

Thus the eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = \lambda_3 = 5$ (that is, the eigenvalue 5 has multiplicity 2).

Now we look for corresponding eigenvectors:

$$A - \lambda_1 I = \begin{bmatrix} 0 & 0 & -2 \\ 2 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus we can take $\mathbf{v}_1 = (-1, 2, 0)$ for an eigenvector corresponding to $\lambda_1 = 4$. Next,

$$A - \lambda_2 I = \begin{bmatrix} -1 & 0 & -2 \\ 4 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus the corresponding eigenspace is spanned by $\mathbf{v}_2 = (0, 1, 0)$ and $\mathbf{v}_3 = (-2, 0, 1)$.

We see that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly idependent; hence the matrix

$$P = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \begin{bmatrix} -1 & 0 & -2 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

is invertible and we have $A = PDP^{-1}$, where D is the diagonal matrix with diagonal entries $4, 5, 5$.

Just to make sure we did not make any mistake, let us check if $AP = PD$:

$$AP = \begin{bmatrix} -4 & 0 & -8 \\ 8 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = PD.$$

So, our calculations are correct. ∎

Section 5.5

Exercise 2: Find the eigenvalues ans a basis for each eigenspace for the following matrix acting on $\mathbb{C}^2$:

$$A = \begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}.$$

Solution. We have

$$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & -5 \\ 1 & 1-\lambda \end{vmatrix} = (5-\lambda)(1-\lambda) + 5 = \lambda^2 - 6\lambda + 10.$$

The discriminat is $D = 6^2 - 40 = -4$, so the roots are $(6 \pm \sqrt{D})/2$, that is, $\lambda_1 = 3 - i$ and $\lambda_2 = 3 + i$.

We have

$$A - \lambda_1 I = \begin{bmatrix} 2+i & -5 \\ 1 & -2+i \end{bmatrix}.$$

We already know that the rows of $A - \lambda_1 I$ are linearly dependent, so to find a solution to $(A - \lambda_1 I)\mathbf{x} = \mathbf{0}$, it is enough to satisfy the first equation, which reads $(2+i)x_1 - 5x_2 = 0$. We can take $x_1 = 5$ so that x_2 looks simpler. Then we have $x_2 = 2 + i$. Hence, the corresponding eigenspace is spanned by

$$\mathbf{v}_1 = \begin{bmatrix} 5 \\ 2+i \end{bmatrix}.$$

Likewise, to solve $(A - \lambda_2 I)\mathbf{x} = \mathbf{0}$, we have to solve the first equation, which is $(2-i)x_1 - 5x_2 = 0$. We take $x_1 = 5$ and $x_2 = 2 - i$ and obtain an eigenvector

$$\mathbf{v}_1 = \begin{bmatrix} 5 \\ 2-i \end{bmatrix}.$$

∎