Section 1.1

Exercise 10: We are given the following augumented matrix

$$\begin{bmatrix} 1 & -2 & 0 & 3 & -2 \\ 0 & 1 & 0 & -4 & 7 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}.$$

We have to bring it to the diagonal form. The entries below the diagonal are already zero, so we work from bottom to top. Adding the fourth row with coefficients -3 and 4 to the first and second rows, we obtain

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}.$$

Column 3 is fine, so it remains to add Row 2 multiplied by 2 to Row 1. We obtain

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}.$$

Thus $(-3, -5, 6, -3)$ is the unique solution. ∎

Exercise 26: The obvious augmented matrix with the unique solution $(-2, 1, 0)$ is

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Possible matrices can be obtained by applying one or a few row operation to the above matrix. Here are two more examples:

$$\begin{bmatrix} 3 & 0 & 0 & -6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 & 0 & 0 & -6 \\ 0 & 1 & 0 & 1 \\ 3 & 0 & 1 & -6 \end{bmatrix}.$$

∎

Section 1.2

Exercise 4: We have to reduce the matrix

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix}$$

to reduced echelon form. We take the first non-zero column which is Column 1 and choose the entry 1 for pilot. Using row replacement, we create zeros below 1:

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -34 \end{bmatrix}.$$

Now (Step 4) we ignore Row 1 and Column 1. The entry -4 is the pivot now:

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & -10 \end{bmatrix}.$$

The matrix is in echelon form now. Its pivots are the entries 1, -4, and -10, located in Columns 1, 2, and 4 respectively. Now we start Step 5. First we multiply Row 3 by $-1/10$ and eliminate non-zero entries above this pivot:

$$\begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & -4 & -8 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Next, we multiply the second row by $-1/4$ and reduce the entry above the pivot to zero:

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The matrix is in reduced echelon form now. ▮

Exercise 8: We have to find the general solution, given the following augmented matrix:

$$\begin{bmatrix} 1 & 4 & 0 & 7 \\ 1 & 7 & 0 & 10 \end{bmatrix}.$$

We bring it first to echelon form by adding Row 1 multiplied by -2 to Row 2.

$$\begin{bmatrix} 1 & 4 & 0 & 7 \\ 0 & -1 & 0 & -4 \end{bmatrix}.$$

Next, we bring the matrix to reduced echelon form. We have to make leading entries to be equal to 1 by multiplying the second row by -1 and then eliminate the 4 above it:

$$\begin{bmatrix} 1 & 0 & 0 & -9 \\ 0 & 1 & 0 & 4 \end{bmatrix}.$$

We see that the system has infinitely many solutions: $x_1 = -9$, $x_2 = 4$, x_3 is free. ∎

Section 1.3

Exercise 18: The question, if rephrased, asks for which h the system of equations corresponding to the augmented matrix

$$\begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ -2 & 8 & -3 \end{bmatrix}$$

is consistent. We convert the matrix to echelon form pivoting on the first column and then on the second column:

$$\begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 2 & 2h-3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 0 & 2h+7 \end{bmatrix}.$$

Hence, $\mathbf{y}$ is in the span of $\mathbf{v}_1$ and $\mathbf{v}_2$ if and only if $h = -7/2$. ∎

Section 1.4

Exercise 16: Let

$$A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

We have find for which $\mathbf{b}$ the system $A\mathbf{x} = \mathbf{b}$ has a solution. We start with augmented matrix and bring to echelon form:

$$\begin{bmatrix} 1 & -3 & -4 & b_1 \\ -3 & 2 & 6 & b_2 \\ 5 & -1 & -8 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & b_1 \\ 0 & -7 & -6 & 3b_1 + b_2 \\ 0 & 14 & 12 & -5b_1 + b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & b_1 \\ 0 & -7 & -6 & 3b_1 + b_2 \\ 0 & 0 & 0 & 2(3b_1 + b_2) - 5b_1 + b_3 \end{bmatrix}.$$

We can choose b_1, b_2, b_3 so that $2(3b_1 + b_2) - 5b_1 + b_3 = b_1 + 2b_2 + b_3$ is non-zero. (For example, $b_1 = b_2 = 0$ and $b_3 = 1$.) So, the columns of A do not span $\mathbb{R}^3$. The set of $\mathbf{b}$ for which the system $A\mathbf{x} = \mathbf{b}$ has a solution consists of those triples with $b_1 + 2b_2 + b_3 = 0$. ∎

Exercise 32: It is intuitively clear that 3 vectors cannot span $\mathbb{R}^4$ but let argue using theorems from the course. Let the vectors be $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \mathbb{R}^4$. Let A be the 4×3-matrix $[\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$. To test if these vectors span $\mathbb{R}^4$ we have to test whether the system $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^4$. The latter happens if and only if A has pivots in every row. But the number of pivots in A is at most 3 (the number of columns), so we cannot have a pivot in every row. ∎

Section 1.5

Exercise 12: We have to describe all solutions of $A\mathbf{x} = \mathbf{0}$ in vector parametric form, where

$$A = \begin{bmatrix} 1 & 5 & 2 & -6 & 9 & 0 \\ 0 & 0 & 1 & -7 & 4 & -8 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We bring A to reduced echelon form:

$$A \sim \begin{bmatrix} 1 & 5 & 2 & -6 & 9 & 0 \\ 0 & 0 & 1 & -7 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 0 & 8 & 1 & 0 \\ 0 & 0 & 1 & -7 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus x_2, x_4, x_5 are the free variables. The general solution of $A\mathbf{x} = \mathbf{0}$ has the form

$$\begin{aligned} x_1 &= -5x_2 - 8x_4 - x_5, \\ x_3 &= 7x_4 - 4x_5 \\ x_6 &= 0. \end{aligned}$$

We can rewrite it as

$$\mathbf{x} = x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -8 \\ 0 \\ 7 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ -4 \\ 0 \\ 1 \\ 0 \end{bmatrix} = x_2\mathbf{u} + x_4\mathbf{v} + x_5\mathbf{w}.$$

Thus $\mathbf{x} = r\mathbf{u} + s\mathbf{v} + t\mathbf{w}$ (where we replaced the free variables by parameters r, s, t) is the required parametric vector form. ∎

Section 1.6

Exercise 8: We have to find coefficents in the following chemical reaction:

$$(x_1)\, KMnO_4 + (x_2)\, MnSO_4 + (x_3)\, H_2O \to (x_4)\, MnO_2 + (x_5)\, K_2SO_4 + (x_6) H_2SO_4.$$

Since there are 5 types of atoms, namely (K, Mn, O, S, H), we use vectors in $\mathbb{R}^5$. We get

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 4 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} = x_4 \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 0 \\ 0 \\ 4 \\ 1 \\ 2 \end{bmatrix}.$$

This corresponds to a matrix equation $A\mathbf{x} = \mathbf{0}$ with

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & -2 & 0 \\ 1 & 1 & 0 & -1 & 0 & 0 \\ 4 & 4 & 1 & -2 & -4 & -4 \\ 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 2 & 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 2 & 0 \\ 0 & 4 & 1 & -2 & 4 & -4 \\ 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 2 & 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 2 & -4 & -4 \\ 0 & 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 2 & 0 & 0 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 2 & -4 & -4 \\ 0 & 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & -4 & 8 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 2 & -4 & -4 \\ 0 & 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 & -4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 & -6 \\ 0 & 0 & 0 & 1 & 0 & -5/2 \\ 0 & 0 & 0 & 0 & 1 & -1/2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -3/2 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -5/2 \\ 0 & 0 & 0 & 0 & 1 & -1/2 \end{bmatrix}$$

Here x_6 is a free variable. Since the last column involves division by 2, let us take $x_6 = 2$. This gives us $\mathbf{x} = (2, 3, 2, 5, 1, 2)$.

Let us make a check. For example, of oxygen atoms we have $2 \cdot 4 + 3 \cdot 4 + 2 \cdot 1 = 22$ in the left-hand side and $5 \cdot 2 + 1 \cdot 4 + 2 \cdot 4 = 22$ on the right-hand side, which is reassuring. ∎

Section 1.7

Exercise 8: We have to check whether the given 3×4-matrix has linearly independent columns. Actually, we can tell that the column must be linearly independent for any matrix A with these dimensions. Indeed, after we bring A to echelon form, there will be at most 3 pivots (at most one per row). So, the corresponding linear homogeneous system $\mathbf{x} = \mathbf{0}$ has free variables and, hence, non-trivial solutions. ∎

Exercise 40: Suppose an $m \times n$ matrix A has n pivot columns. We have to explain why for each $\mathbf{b}$ the equation $A\mathbf{x} = \mathbf{b}$ has at most one solution.

As no two pivots can be in one column, we have exactly one pivot in each column. If we had two different solutions to $A\mathbf{x} = \mathbf{b}$, say $\mathbf{y}$ and $\mathbf{z}$, then we would have $A(\mathbf{z}-\mathbf{y}) = \mathbf{0}$. But the *homegenous* system $A\mathbf{x} = \mathbf{0}$ has no free variable and, therefore, cannot have any non-trivial solutions. ▮

Section 1.8

Exercise 2: We have

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} .5 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & .5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = a \begin{bmatrix} .5 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ .5 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ .5 \end{bmatrix} = \begin{bmatrix} .5a \\ .5b \\ .5c \end{bmatrix}.$$

For $\mathbf{u} = (1, 0, 4)$ we obtain $T(\mathbf{u}) = (.5, 0, -2)$. (Recall that $(x_1, \ldots, x_p)$ is a shorhand notation for the vector whose components are $x_1, \ldots, x_p$.) ▮

Exercise 24: Suppose vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ span $\mathbb{R}^n$, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Suppose that $T(\mathbf{v}_i) = \mathbf{0}$ for $i = 1, \ldots, p$. Show that T is the zero transfomation.

Solution: Take any $\mathbf{x} \in \mathbb{R}^n$. As $\mathbf{v}_1, \ldots, \mathbf{v}_p$ span $\mathbb{R}^n$, we have $\mathbf{x} = c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p$ for some real numbers $c_1, \ldots, c_p$. (This representation may be not unique, but this does not affect our argument.) By the properties of linear transformations, we have

$$T(\mathbf{x}) = T(c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + \ldots + c_pT(\mathbf{v}_p) = \mathbf{0},$$

the last equality being true as $T(\mathbf{v}_i) = \mathbf{0}$ for every i. ▮