Practice problems for Exam 3

1. Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

   (a) Determine whether A is diagonalizable. If so, find a matrix S such that $S^{-1}AS$ is diagonal. If not, explain why not.

   (b) What are the eigenvalues of A^{-1}? Is A^{-1} diagonalizable? Explain.

   (a) A is indeed diagonalizable. Computing the characteristic polynomial, which is $\det(A - \lambda I)$, one finds the eigenvalues to be 1, 1, and 2. By looking at the nullspace of $A - I$, and finding it to be two-dimensional, one concludes that A is indeed diagonalizable. So find three linearly independent eigenvectors and use them to construct S. And if this were an actual problem, you can check that you have a valid diagonalizing matrix S by computing the products AS and SD (where D is a 3×3 matrix with the eigenvalues of A on the diagonal, in an order which conforms to your choice of ordering the eigenvectors), and making sure they are the same ... because, you see, $S^{-1}AS = D$ can only hold if $AS = SD$.

   (b) First one might ask how we know for sure that A^{-1} even exists. Well ... in this case we know it from part (a), because 0 is not an eigenvalue of A, which is true if and only if A is invertible. Now refer to Exercise 5.1.25, wherein one proves that if A is invertible, then the eigenvalues of A^{-1} are the reciprocals of the eigenvalues of A. In the process of proving this, one finds that eigenvectors of A are also eigenvectors of A^{-1}. In fact, the *eigenspaces* of A^{-1} and A are the same ... which makes sense when you think about this geometrically. Consider E_2, for example: The matrix A doubles all vectors in this line, but these vectors don't change direction. So, considering that A^{-1} "undoes" what A does, what will A^{-1} do to the vectors in this line? It will "halve" all of them. So the same line which is the eigenspace E_2 for A is the eigenspace $E_{1/2}$ for A^{-1}. And the plane which remains fixed under the action of A will also remain fixed under the action of A^{-1}.

   At any rate, A^{-1} is indeed diagonalizable, and the eigenvalues are 1, 1, and 1/2.

2. Produce an example of a 2×2 matrix with eigenvalue 1, whose column space is the line $y = -x$ and whose nullspace is the line $y = 2x$.

   The statement of the problem tells you that the eigenvalues are 1 and 0. The fact that the nullspace is nontrivial means that your matrix can't be invertible ... hence, 0 is an eigenvalue. As we have two distinct real eigenvalues, the matrix is diagonalizable. So you should be able to write down a factorized version of your matrix first, as PDP^{-1}, for suitably constructed P and D. Then multiply out the product, and you have a 2×2 matrix with the required properties.

3. Suppose two animal species, slinkies and tinkies, coexist in the same habitat. Suppose s_k is the slinkie population at time k, and t_k is the tinkie population at time k. Let $\vec{x}_0 = \begin{bmatrix} s_0 \\ t_0 \end{bmatrix}$, and assume the initial populations s_0 and t_0 are positive. Write down a dynamical system describing this scenario, which predicts that if $t_0 < 2s_0$, then both species will survive in the long run, but if $t_0 \geq 2s_0$, the slinkies will eventually die out.

I would approach this problem by starting with a drawing of a "trajectory profile" ...something like the figure on page 349. For 2×2 dynamical systems with two real eigenvalues, both positive, and neither equal to 1, we have three possible types of "trajectory profile"; the origin can be an attractor, a repellor, or a saddle. So which do you need here? Start with a picture (which includes the whole plane and not just the first quadrant) of the kinds of trajectories you need. The line $t = 2s$ will have to be a key feature of this picture. From the problem statement, we see that trajectories corresponding to initial states *on* this line will have to head toward the origin (read, both species are doomed). So this line should be an eigenspace corresponding to an eigenvalue between 0 and 1 (any such number will do; this problem has many, many valid solutions). The other eigenspace should be another line. Where should that line be? (There are, in fact, many lines which will work, but not just any slope will do.) And what sort of eigenvalue should the other one be?

Once you have chosen your eigenvalues and eigenspaces, then to get the coefficient matrix for your dynamical system, you will have to write out the full diagonalization factorization and then multiply ...much like what you did in Problem 2.

4. Let s and t be numbers between 0 and 1. Let

$$A = \begin{bmatrix} s & 1-t \\ 1-s & t \end{bmatrix}$$

(a) Compute

$$A \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

What does this computation tell you about A?

(b) Explain why 1 is an eigenvalue of A^T.

(c) Compute $\lim_{k \to \infty} A^k$. *Hint:* Diagonalize A.

(a) It tells you that $s + t - 1$ is an eigenvalue of A, and that $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is a corresponding eigenvector.

(b) This follows from a fact you may or may not have remembered (see Exercise 5.1.29 and, while you're at it, Exercise 5.1.30): If the rows of a matrix all add up to the same number, then that number is an eigenvalue (and in that case, notice that the vector of all ones is a corresponding eigenvector).

(c) For this part, you need to recognize that part (a) tells you that $s+t-1$ is an eigenvalue of A, and that part (b) tells you that 1 is also an eigenvalue of A (because the eigenvalues of A and A^T are the same. You already have an eigenvector corresponding to $s+t-1$; to find an eigenvector corresponding to eigenvalue 1, look at the nullspace of $A-I$. I get the limiting matrix to be

$$\frac{1}{s+t-2}\begin{bmatrix} t-1 & t-1 \\ s-1 & s-1 \end{bmatrix}$$

(I hope that's correct.)

5. This is a three-part question; the three parts are related.

   (a) Suppose A is an $m \times n$ matrix with linearly independent rows; suppose $\vec{y}, \vec{z} \in \mathbf{R}^m$, and suppose $A^T\vec{y} = A^T\vec{z}$. Show that $\vec{y} = \vec{z}$.

   (b) Suppose $A = BC$, where

$$B = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} \\ -1/\sqrt{2} & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

   Show that if $A\vec{x} = \vec{b}$ is an inconsistent system, then finding the least squares solution(s) to this system is equivalent to solving the system $C\vec{x} = B^T\vec{b}$.

   (c) Let A, B, and C be as in part (b). Find all least squares solutions to the system

$$A\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

(a) Rewrite $A^T\vec{y} = A^T\vec{z}$ as $A^T(\vec{y} - \vec{z}) = \vec{0}$. This is a homogeneous system, and since the columns of A^T are linearly independent, the only solution is $\vec{0}$. Therefore $\vec{y} - \vec{z} = \vec{0}$.

(b) Start with $A^TA\vec{x} = A^T\vec{b}$, and substitute BC for A. Noting that B has orthonormal columns and applying part (a) appropriately, you can get the result.

(c) The point here is that you can use part (b), which enables you to avoid multiplying B and C together.

6. Let
$$\vec{v_1} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v_2} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}, \quad \vec{v_3} = \begin{bmatrix} -2 \\ -5 \\ 13 \end{bmatrix}$$
   (a) Show that $\vec{v_1}, \vec{v_2}, \vec{v_3}$ form a basis for $\mathbf{R}^3$.
   (b) Give a formula for the linear transformation $S : \mathbf{R}^3 \to \mathbf{R}^3$ which projects vectors onto the plane spanned by $\vec{v_1}$ and $\vec{v_2}$.

(a) It's enough to verify that this is a mutually orthogonal triplet.

(b) I just want to point out here that the problem does not ask you for a matrix; you are only asked for a formula, in which vectors are inputted, and vectors are outputs.

7. Let $A = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}$. Determine $\lim_{n\to\infty} A^n$.

This is somewhat like Problem 4, but simpler. Diagonalize A so that you can take nth powers of the diagonal matrix, instead of nth powers of A itself.

8. Let A be a 4×4 invertible matrix, and suppose B is its inverse.
   (a) Establish that the rows of B are linearly independent.
   (b) Let $V = \text{span}(\vec{B}^1, \vec{B}^2)$. What is the dimension of V? Justify your answer.
   (c) Find a basis for $V^\perp$. Prove that the set of vectors you choose really is a basis for $V^\perp$.

(a) Well, B is the inverse of A, so B is also invertible. And matrices are invertible if and only if they have linearly independent columns, and if and only if they have linearly independent rows.

(b) Taking all four rows of B gives you a linearly independent set of vectors; therefore, any subset of these is also a linearly independent set. Therefore the pair $\vec{B}^1, \vec{B}^2$ really is a basis for V, and this gives V dimension 2.

(c) So here's the tricky part of the problem. You know that, since we're working in $\mathbf{R}^4$, and since $\dim V = 2$, the dimension of $V^\perp$ is also 2. So you know you're looking for two vectors to form a basis. You need two vectors $\vec{y}$ and $\vec{z}$ such that $\vec{y}$ is orthogonal to $\vec{B}^1$ and to $\vec{B}^2$, and $\vec{z}$ is orthogonal to $\vec{B}^1$ and to $\vec{B}^2$. Have we used the fact that A and B are inverses? Yes, but ... we only used it a little bit, in part (a). Think about the equation $BA = I$. On the left is a matrix product. How are the individual entries of this matrix product computed?

9. Let

$$V = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right)$$

Let $T : \mathbf{R}^4 \to \mathbf{R}^4$ be orthogonal projection onto V. Find the matrix A which represents T.

First derive a formula like you did in 6(b). Then remember that to construct the matrix which represents T, you should apply T to each of the four standard unit vectors.

10. Let $V = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$, where

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 4 \\ 4 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} -1 \\ -3 \\ 0 \\ 8 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} 2 \\ -16 \\ 3 \\ 1 \end{bmatrix}$$

Find an orthonormal basis for V.

This is just a straightforward application of the Gram-Schmidt algorithm, but *remember, if you're really taking an exam, you should check that the vectors you end up with really are mutually orthogonal.* You don't need to write that part down, but you should visually inspect your vectors, and check mutual orthogonality in your head. Don't forget to normalize the vectors at the end; also remember normalizing is easier if you scale to get rid of all fractions first.

11. Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Find an orthonormal basis for the column space of A.

If you perform Gaussian elimination on A (which requires one measly step), you find the pivot columns to be $\vec{A}_1, \vec{A}_2$, and $\vec{A}_4$. So extract those three vectors (from the original matrix!) as a basis, and apply Gram-Schmidt to get an alternate basis consisting of mutually orthogonal vectors. Don't forget to verify mutual orthogonality and to normalize.