The game chromatic number of random graphs

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Abstract

Given a graph G and an integer k, two players take turns coloring the vertices of G one by one using k colors so that neighboring vertices get different colors. The first player wins iff at the end of the game all the vertices of G are colored. The game chromatic number $\chi_g(G)$ is the minimum k for which the first player has a winning strategy. In this paper we analyze the asymptotic behavior of this parameter for a random graph $G_{n,p}$. We show that with high probability the game chromatic number of $G_{n,p}$ is at least twice its chromatic number but, up to a multiplicative constant, has the same order of magnitude. We also study the game chromatic number of random bipartite graphs.

1 Introduction

Let $G = (V, E)$ be a graph and let k be a positive integer. Consider the following game in which two players Maker and Breaker take turns coloring the vertices of G with k colors. Each move consists of choosing an uncolored vertex of the graph and assigning to it a color from $\{1, \ldots, k\}$ so that resulting coloring is *proper*, i.e., adjacent vertices get different colors. Maker wins if all the vertices of G are eventually colored. Breaker wins if at some point in the game the current partial coloring cannot be extended to a complete coloring of G, i.e., there is an uncolored vertex such that each of the k colors appears at least once in its neighborhood. We assume that Maker goes first (our results will not be sensitive to this choice). The *game chromatic number* $\chi_g(G)$ is the least integer k for which Maker has a winning strategy.

This parameter is well defined, since it is easy to see that Maker always wins if the number of colors is larger than the maximum degree of G. Clearly, $\chi_g(G)$ is at least as large as the ordinary chromatic number $\chi(G)$, but it can be considerably more. For example, let G be a complete bipartite graph $K_{n,n}$ minus a perfect matching M and consider the following strategy for Breaker. If Maker colors vertex v with color c then Breaker responds by coloring the vertex u matched with v in the matching M with the same color c. Note that now c cannot be used on any other vertex in the graph. Therefore, if the number of colors is less than n, Breaker wins the game. This shows that there are bipartite

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graphs with arbitrarily large game chromatic number and thus there is no upper bound on $\chi_g(G)$ as a function of $\chi(G)$.

The game was first considered by Brams about 25 years ago in the context of coloring planar graphs and was described in Martin Gardner's column [10] in Scientific American in 1981. The game remained unnoticed by the graph-theoretic community until Bodlaender [3] re-invented it. It has been studied for various classes of graphs in recent years. Faigle, Kern, Kierstead and Trotter [9] proved that the game chromatic number of a forest is at most 4, and that there are forests which require that many colors. The game chromatic number of planar graphs was studied by Kierstead and Trotter [13], who showed that for such graphs the game chromatic number is at most 33. Moreover they proved that any graph embeddable on an orientable surface of genus q has game chromatic number bounded by a function of q. Several additional results on χ_g and some related parameters were obtained in [4, 7, 8, 17, 12, 16, 6, 14]. For a recent survey see Bartnicki, Grytczuk, Kierstead and Zhu [2].

In this paper, we study the game chromatic number of the random graph $G_{n,p}$. As usual, $G_{n,p}$ stands for the probability space of all labeled graphs on n vertices, where every edge appears independently with probability $p = p(n)$. We assume throughout the paper that the edge probability $p \leq 1-\eta$, where $\eta > 0$ is an arbitrarily small, but fixed, constant. Define $b = \frac{1}{1-p}$ and note that $\log_b x = \frac{\log x}{\log b} = (1+o(1))\frac{\log x}{p}$ for all $x \geq 1$ and $p = o(1)$. Our first result determines the order of magnitude of the game chromatic number of $G_{n,p}$.

Theorem 1.1

(a) *There exists $K > 0$ such that for $\varepsilon > 0$ and $p \geq (\log n)^{K\varepsilon^{-3}}/n$ we have that* **whp**[1]

$$\chi_g(G_{n,p}) \geq (1-\varepsilon)\frac{n}{\log_b np}\,.$$

(b) *If $\alpha > 2$ is a constant, $K = \max\{\frac{2\alpha}{\alpha-1}, \frac{\alpha}{\alpha-2}\}$ and $p \geq (\log n)^K/n$ then* **whp**

$$\chi_g(G_{n,p}) \leq \alpha\frac{n}{\log_b np}\,.$$

It is natural to compare our bounds with the asymptotic behavior of the ordinary chromatic number of random graph. It is known by the results of Bollobás [5] and Łuczak [15]) that **whp** $\chi(G_{n,p}) = (1+o(1))\frac{n}{2\log_b np}$. Thus our result shows that the game chromatic number of $G_{n,p}$ is at least twice its chromatic number, but up to a multiplicative constant has the same order of magnitude.

As already mentioned, there are graphs whose game chromatic number is much larger then the ordinary one. Our next theorem provides the existence of a large collection of such graphs. Let $B_{n,p}$ denote the random bipartite graph with two parts of n vertices where each of the n^2 possible edges appears randomly and independently with probability p. We obtain the following bounds on the game chromatic number of this graph.

[1]A sequence of events $\mathcal{E}_n$ occurs *with high probability* (**whp**) if $\lim_{n\to\infty} \mathbb{P}(\mathcal{E}_n) = 1$

Theorem 1.2

(a) *If $p \geq 2/n$ then*

$$\chi_g(B_{n,p}) \geq \frac{n}{10(\log n)(\log_b np)}.$$

(b) *If $\alpha > 2$ is a constant, $K = \max\{\frac{2\alpha}{\alpha-1}, \frac{\alpha}{\alpha-2}\}$ and $p \geq (\log n)^K/n$ then* **whp**

$$\chi_g(B_{n,p}) \leq \alpha \frac{n}{\log_b np}.$$

The rest of this paper is organized as follows. The next two sections contain proofs of lower and upper bounds in Theorem 1.1. In Section 4 we consider the game chromatic number of random bipartite graphs and prove Theorem 1.2. The last section of the paper contains some concluding remarks and open problems.

Unless the base is specifically mentioned, log will refer to natural logarithms. We often refer to the following Chernoff-type bounds for the tails of binomial distributions (see, e.g., [1] or [11]). Let $X = \sum_{i=1}^{n} X_i$ be a sum of independent indicator random variables such that $\mathbb{P}(X_i = 1) = p_i$ and let $p = (p_1 + \cdots + p_n)/n$. Then

$$\begin{aligned} \mathbb{P}(X \leq (1-\varepsilon)np) &\leq e^{-\varepsilon^2 np/2}, \\ \mathbb{P}(X \geq (1+\varepsilon)np) &\leq e^{-\varepsilon^2 np/3}, \qquad \varepsilon \leq 1, \\ \mathbb{P}(X \geq \mu np) &\leq (e/\mu)^{\mu np}. \end{aligned}$$

2 Lower bound on the game chromatic number of $G_{n,p}$

Suppose that $p \geq (\log n)^{K\varepsilon^{-3}}/n$, where K is a sufficiently large constant, and that the number of colors k satisfies $k \leq (1-\varepsilon)\frac{n}{\log_b np}$. We begin by defining a series of numbers (which will serve as cut-offs for Breaker's strategy). Let

$$\ell_1 = \log_b n - \log_b \log_b np - 10 \log_b \log n$$

and note that ℓ_1 has been chosen so that we have

$$(1-p)^{\ell_1} = \frac{\log_b np (\log n)^{10}}{n} = (1+o(1))\frac{\ell_1 (\log n)^{10}}{n}. \tag{1}$$

Set

$$\ell_2 = \frac{\varepsilon \ell_1}{20} \qquad \text{and} \qquad \ell_3 = \frac{\varepsilon^3 \ell_1}{2 \cdot 10^6}.$$

For $S \subseteq [n]$ let $N(S)$ be the neighbors of S which are not in S and let $\overline{N}(S) = [n] \setminus (S \cup N(S))$.

We are now ready to describe Breaker's strategy. Fix a color i. Whenever Maker uses i, Breaker will respond by using color i in his next move. At the beginning Breaker chooses this vertex arbitrarily; only when ℓ_1 vertices are colored i will Breaker choose carefully the next vertex to color. Let T denote the set of uncolored vertices in $\overline{N}(C_i)$ at the time when the set of vertices C_i that have been colored with i satisfies $|C_i| = \ell_1$. At this point Breaker identifies a maximum size independent subset I_1 of T. When Breaker next uses color i, he will color the vertex $v \in T$ which has as many neighbors in I_1 as possible. After this, $I_1 \leftarrow I_1 \setminus N(v)$. When $|I_1| \leq \ell_3$ we say that Breaker has completed *elimination iteration 1.* After completing elimination iteration j, Breaker will start a new iteration by identifying the largest independent set I_{j+1} in the set of uncolored vertices in the current $\overline{N}(C_i)$ and continue with the previous strategy. This continues as long as at the start of a new iteration the set $\overline{N}(C_i)$ contains an independent set of uncolored vertices of size at least ℓ_2. Once $\overline{N}(C_i)$ does not contain any more independent sets of size ℓ_2, from then on Breaker again colors arbitrarily with i when desired.

In order to validate Breaker's strategy we must establish a few facts. We begin by considering the size of $\overline{N}(C_i)$ when $|C_i| = \ell_1$.

Lemma 2.1 *For every subset $S \subseteq [n]$ of size $|S| = \ell_1$* **whp** $\ell_1(\log n)^9 \leq |\overline{N}(S)| \leq \ell_1(\log n)^{11}$.

Proof. Fix S with $|S| = \ell_1$. The size of $\overline{N}(S)$ is distributed as the binomial $B(n - \ell_1, (1-p)^{\ell_1})$. Therefore by (1) the expected size of $\overline{N}(S)$ is $(1 + o(1))\ell_1(\log n)^{10}$. Thus, it follows from the Chernoff bounds that

$$\mathbb{P}\Big[\exists S:\ |S| = \ell_1,\ |\overline{N}(S)| \notin [\ell_1(\log n)^9, \ell_1(\log n)^{11}]\Big] \leq 2\binom{n}{\ell_1}e^{-\Theta(\ell_1(\log n)^{10})} = o(1).$$

$\square$

Next we note that if the number of uncolored vertices in $\overline{N}(C_i)$ is sufficiently large, then Breaker should be able to choose a vertex that reduces the size of I_j by a substantial amount.

Lemma 2.2 **Whp** *there do not exist $S, A, B \subseteq [n]$ such that*

1. $|S| = \ell_1$, $a = |A| \in [\ell_3, 3\ell_1]$, $|B| \geq b_1 = 100\varepsilon^{-1}\ell_1(\log n)^2$.
2. $A, B \subseteq \overline{N}(S)$ *and* $A \cap B = \emptyset$.
3. *Every* $x \in B$ *has fewer than* $ap/2$ *neighbors in* A.

Proof. Applying Lemma 2.1 to bound the size of $\overline{N}(S)$ and using $\ell_3 p = \Omega(K \log\log n)$ we see that the probability of this event is at most

$$\begin{aligned} & o(1) + \binom{n}{\ell_1}\sum_{a=\ell_3}^{3\ell_1}\binom{\ell_1(\log n)^{11}}{a}\binom{\ell_1(\log n)^{11}}{b_1}\mathbb{P}(B(a,p) \leq ap/2)^{b_1} \\ \leq\ & o(1) + n^{\ell_1}\sum_{a=\ell_3}^{3\ell_1}(\log n)^{11(a+b_1)}e^{-ab_1p/8} \\ =\ & o(1). \end{aligned}$$

$\square$

Now we consider the number of elimination iterations. Define

$$\ell_2' = \frac{\varepsilon \ell_1}{21},$$

and note that we have $\ell_2' < \ell_2 - \ell_3$. Each elimination iteration removes at least ℓ_2' vertices from the set of vertices that can be colored with color i. Note further that these sets are disjoint and that each forms an independent set in our graph.

Lemma 2.3 Whp *there do not exist* $S, T_1, T_2, \ldots, T_{a_1} \subseteq [n]$, $a_1 = 2000\varepsilon^{-2}$ *such that*

1. $S, T_1, T_2, \ldots, T_{a_1}$ *are pair-wise disjoint independent sets.*
2. $|S| = \ell_1$.
3. $|T_i| = \ell_2'$, $i = 1, 2, \ldots, a_1$.
4. $N(S) \cap T_i = \emptyset$, $i = 1, 2, \ldots, a_1$.

Proof. Let $\mathcal{E}_1$ be the event that such a collection of sets exists. Then, using $\ell_1/\ell_2' = \Theta(1)$, $\frac{\ell_1}{n} = O\left(\frac{\log n}{np}\right) \ll \left(\log n\right)^{-K\varepsilon^{-3}/2}$ together with (1) and Lemma 2.1, we have

$$\begin{aligned}
\mathbb{P}(\mathcal{E}_1) &\leq o(1) + \binom{n}{\ell_1} \binom{\ell_1 (\log n)^{11}}{\ell_2'}^{a_1} (1-p)^{a_1 \binom{\ell_2'}{2}} \\
&\leq o(1) + \left(\frac{ne}{\ell_1}\right)^{\ell_1} \left(\frac{e\,\ell_1 (\log n)^{11}}{\ell_2'} (1-p)^{(\ell_2'-1)/2} \right)^{a_1 \ell_2'} \\
&\leq o(1) + \left(\frac{ne}{\ell_1}\right)^{\ell_1} \left((\log n)^{12} \left(\frac{\ell_1 (\log n)^{10}}{n} \right)^{\varepsilon/42} \right)^{a_1 \varepsilon \ell_1/21} \\
&\leq o(1) + \left(\frac{n}{\ell_1}\right)^{\ell_1} \left(\frac{\ell_1}{n}\right)^{a_1 \varepsilon^2 \ell_1/1000} \left(\log n\right)^{a_1 \varepsilon \ell_1} \\
&\leq o(1) + \left(\frac{\ell_1}{n} \left(\log n\right)^{2000/\varepsilon} \right)^{\ell_1} \\
&= o(1).
\end{aligned}$$

□

We will complete the proof by showing that most colors are used on roughly ℓ_1 vertices. We will use the following Lemma to bound the number of colors that are used on significantly more vertices. We define a fourth cut-off

$$\ell_0 = \ell_1 + \frac{12\ell_1}{\ell_3 p} \cdot a_1.$$

Note that $a_1 \ell_1/\ell_3$ is a constant and $(1-p)^{\ell_0} = \Omega\left((1-p)^{\ell_1}\right)$.

Lemma 2.4 Whp *there do not exist pair-wise disjoint sets* $S_1, S_2, \ldots, S_{b_2}, U$, $b_2 = \frac{n}{\ell_1 (\log n)^7}$ *such that*

1. $|S_i| = \ell_0$ *for* $i = 1, 2, \ldots, b_2$.

2. $|U| = \lceil n/\log n \rceil$.

3. $|U \cap \overline{N}(S_i)| \le \ell_1 (\log n)^8$ *for* $i = 1, 2, \ldots, b_2$.

Proof. Let $\mathcal{E}_2$ be the event that such a collection of sets exists. For every choice of S_i and U the size $|U \cap \overline{N}(S_i)|$ is distributed as the binomial $B(n/\log n, (1-p)^{\ell_0})$ with expectation $(n/\log n)(1-p)^{\ell_0} = \Omega\big(n(1-p)^{\ell_1}/\log n\big) = \Omega(\ell_1(\log n)^9)$. Thus, by the Chernoff bound,

$$\begin{aligned}\mathbb{P}(\mathcal{E}_2) &\le \binom{n}{n/\log n}\left(\binom{n}{\ell_0}\mathbb{P}\big(B(n/\log n, (1-p)^{\ell_0}) \le \ell_1 \log^8 n\big)\right)^{b_2} \\ &\le n^{n/\log n} n^{2b_2\ell_1} e^{-\Omega(b_2\ell_1(\log n)^9)} \\ &= o(1).\end{aligned}$$

□

We now use these Lemmas to complete the proof, assuming that the associated low probability events do not occur. Assume for the sake of contradiction that the game reaches the point where only $n/\log n$ vertices remain to be colored. Let U be the set of uncolored vertices. Let C_i denote the set of vertices colored i at this point and let $c_i = |C_i|$ for $i = 1, 2, \ldots, k$. Observe that **whp**

$$c_i \le (2 + .01\varepsilon)\ell_1,\ i = 1, 2, \ldots, k \tag{2}$$

since the right hand side is an upper bound on the size of an independent set in $G_{n,p}$.

Claim 2.5 *Let i be a color such that $c_i \ge (1+\varepsilon/4)\ell_1$. If C_i' are the first ℓ_0 vertices to be colored with color i then we have*

$$\big|\overline{N}(C_i') \cap U\big| \le b_1 = \frac{100\ell_1(\log n)^2}{\varepsilon}$$

Proof.

Assume for the sake of contradiction that $|\overline{N}(C_i') \cap U| > b_1$. Let S_t be the set of vertices which are colored i at time t. At all times t such that $|S_t| < \ell_0$ the set of uncolored vertices in $\overline{N}(S_t)$ has size at least $|\overline{N}(C_i') \cap U| > b_1$. Let I_j be the independent set that is being eliminated at time t. Since $|I_j|$ is smaller than the independence number of $G_{n,p}$, Lemma 2.2 implies that Breaker can choose a vertex that eliminates at least $|I_j|p/2$ vertices from I_j. Therefore, each elimination iteration involves at most $2 \cdot |I_j|/(\ell_3 p/2) < 9\ell_1/(\ell_3 p)$ uses of color i. But Lemma 2.3 implies that there are at most a_1 elimination iterations. Therefore, Breaker will complete all of the elimination iterations before color i has been used ℓ_0 times. Note that after the elimination process is completed one can only color at most $\ell_2 = \varepsilon\ell_1/20$ vertices by color i. Therefore,

$$c_i < \ell_1 + 9a_1\ell_1/(\ell_3 p) + \varepsilon\ell_1/20 < (1+\varepsilon/4)\ell_1.$$

This is a contradiction. □

It follows from Claim 2.5 and Lemma 2.4 that there at most $b_2 = \frac{n}{\ell_1(\log n)^7}$ colors i such that $c_i > (1+\varepsilon/2)\ell_1$. Applying this fact together with (2) and $k \le (1-\varepsilon)n/\ell_1$ we obtain

$$n - \frac{n}{\log n} = \sum_{i=1}^{k} c_i \leq b_2(2+.01\varepsilon)\ell_1 + (k-b_2)(1+\varepsilon/4)\ell_1 < (1-\varepsilon/2)n.$$

This is a contradiction.

3 Upper bound on the game chromatic number of $G_{n,p}$

Let α be any constant greater than 2, $K > \max\{\frac{2\alpha}{\alpha-1}, \frac{\alpha}{\alpha-2}\}$, $p > (\log n)^K/n$ and let the number of colors be $k = \alpha\frac{n}{\log_b np}$. We begin with Maker's strategy. Let $\mathcal{C} = (C_1, C_2, \dots, C_k)$ be a collection of pair-wise disjoint sets. Let $\bigcup\mathcal{C}$ denote $\bigcup_{i=1}^{k} C_i$. For a vertex v let

$$A(v,\mathcal{C}) = \{i \in [k] :\ v \text{ is not adjacent to any vertex of } C_i\}\,.$$

and set

$$a(v,\mathcal{C}) = |A(v,\mathcal{C})|.$$

Note that $A(v,\mathcal{C})$ is the set of colors that are available at vertex v when the partial coloring is given by the sets in $\mathcal{C}$ and $v \notin \bigcup\mathcal{C}$. Maker's strategy can now be easily defined. Given the current color classes $\mathcal{C}$, Maker chooses an uncolored vertex v with the smallest value of $a(v,\mathcal{C})$ and colors it by any available color.

In order to establish that Maker's strategy succeeds **whp**, we consider a sequence of landmarks in the play of the game. As the game evolves, we let u denote the number of uncolored vertices in the graph. So, we think of u as running 'backward' from n to 0. Below we define a sequence of thresholds $d_0 \geq d_1 \geq \cdots \geq d_{r+1}$ and consider the 'times' u_i which are defined to be the last times (i.e. minimum value of u) for which Maker colors a vertex for which there are at least d_i available colors.

We begin with the first landmark, u_0. Let

$$\beta = k\cdot(np)^{-1/\alpha} = \alpha\frac{n\,(np)^{-1/\alpha}}{\log_b np}\,, \qquad\qquad \gamma = \frac{10n\log n}{\beta}$$

and

$$B(\mathcal{C}) = \{v :\ a(v,\mathcal{C}) < \beta/2\}\,.$$

We begin by showing that with high probability every coloring of the full vertex set has the property that there are at most γ vertices with less than $\beta/2$ available colors.

Lemma 3.1 **Whp**, *for all collections* $\mathcal{C}$,

$$|B(\mathcal{C})| \leq \gamma.$$

Proof. Fix $\mathcal{C}$. Then for every $v \notin \bigcup\mathcal{C}$, the number of colors available at v is the sum of independent indicator variables X_i, where $X_i = 1$ if v has no neighbors in C_i. Then $\mathbb{P}(X_i = 1) = (1-p)^{|C_i|}$ and since $(1-p)^t$ is a convex function we have

$$\begin{aligned}\mathbf{E}(a(v,\mathcal{C})) &= \sum_{i=1}^{k}(1-p)^{|C_i|}\\ &\geq k(1-p)^{(|C_1|+\cdots+|C_k|)/k}\\ &\geq k(1-p)^{n/k}=\beta.\end{aligned}$$

It follows from the Chernoff bound that

$$\mathbb{P}(a(v,\mathcal{C})\leq \beta/2)\leq e^{-\beta/8}.$$

Thus,

$$\mathbb{P}\big(\exists\mathcal{C}\text{ with }|B(\mathcal{C})|>\gamma\big)\leq k^n\binom{n}{\gamma}e^{-\beta\gamma/8}=o(1).$$

□

Set u_0 to be the last time for which Maker colors a vertex with at least $d_0=\beta/2$ available colors, i.e.,

$$u_0=\min\left\{u:\ a(v,\mathcal{C}_u)\geq d_0=\beta/2,\ \text{for all } v\notin\bigcup\mathcal{C}_u\right\},$$

where $\mathcal{C}_u$ denotes the collection of color classes when u vertices remain uncolored. It follows from Lemma 3.1 that **whp** $u_0\leq\gamma$ (we apply the Lemma to the final coloring). This implies that at some point where the number of uncolored vertices is less than γ, every vertex still has at least $d_0=\beta/2$ available colors. In particular, if $\beta/2>\gamma$ (this happens, e.g., for constant p and $\alpha>2$) we see that Maker wins the game since no vertex will ever run out of colors. On the other hand, the proof that Maker's strategy succeeds also for $p=o(1)$ needs more delicate arguments, which we present next.

Since u_0 is both defined and understood, we are ready to define $u_1,\ldots,u_{r+1}$. These landmarks are defined in terms of thresholds $d_1,\ldots,d_{r+1}$, where d_i is a lower bound on the number of colors available at every uncolored vertex (note that $d_0=\beta/2$ was set above). We set $d_{i+1}=d_i-x_i$ where the x_i's will be defined below and

$$u_i=\min\left\{u:\ a(v,\mathcal{C}_u)\geq d_i,\ \text{for all } v\notin\bigcup\mathcal{C}_u\right\}$$

We will choose the x_i's so that

$$u_i\geq\log\log n\quad\text{implies}\quad u_{i+1}\leq\frac{u_i}{10}. \tag{3}$$

We define r (and hence establish the end of our series of landmarks) by $u_r\geq\log\log n>u_{r+1}$. Note that if (3) holds then we have

$$r\leq\log n.$$

We will also ensure that we have

$$\sum_{i=0}^{r}x_i\leq d_0/2. \tag{4}$$

If we can choose the x_i's so that (3) and (4) are satisfied then Maker will succeed in winning the game. Indeed, when there are u_r uncolored vertices, there are fewer than $10\log\log n$ vertices to color and the lists of available colors at these vertices have size at least

$$d_0 - \sum_{i=0}^{r} x_i \geq d_0/2 > \beta/4 > 10\log\log n, \tag{5}$$

where the lower bound follows from the facts that $\beta \geq \Omega\big((np)^{\frac{\alpha-1}{\alpha}}/\log np\big)$, $np > (\log n)^K$ and $K > \frac{2\alpha}{\alpha-1}$.

The key to our analysis (and the choice of x_i's) is the following observation. Between the point when there are u_i uncolored vertices and the end of the game every vertex in $[n] \setminus \bigcup \mathcal{C}_{u_{i+1}}$ must lose at least $d_i - d_{i+1} = x_i$ of its available colors. Indeed, such a vertex v must have at least d_i available colors when there are u_i vertices uncolored but has less then d_{i+1} available colors when v itself is colored. This implies that the graph induced on $[n] \setminus \bigcup \mathcal{C}_i$ has at least $u_{i+1}x_i$ edges. Before we proceed, we need another technical Lemma, which bounds the number of edges spanned by subsets of $G_{n,p}$. For each positive integer s define

$$\phi = \phi(s) = (5ps + \log n)s.$$

Lemma 3.2 **Whp** *every subset S of $G_{n,p}$ of size s spans at most $\phi = \phi(s) = (5ps + \log n)s$ edges.*

Proof.

$$\begin{aligned}
\mathbb{P}\big(\exists\, S \text{ with } e(S) > \phi\big) &\leq \sum_{s=2}^{n} \binom{n}{s} \binom{\binom{s}{2}}{\phi} p^{\phi} \\
&\leq \sum_{s=2}^{n} \left(\frac{ne}{s}\left(\frac{e}{10}\right)^{\log n}\right)^{s} \\
&= o(1).
\end{aligned}$$

□

We henceforth assume that the low probability events given in Lemmas 3.1 and 3.2 do not occur. It follows from our key observation that we have

$$x_i u_{i+1} \leq \phi(u_i) = (5pu_i + \log n)u_i. \tag{6}$$

Thus to achieve (3) it suffices to take

$$x_i \geq 10(5pu_i + \log n).$$

But, since $u_i \leq u_0/10^i$ and $u_0 \leq \gamma$, we can take

$$x_i = \frac{5p\gamma}{10^{i-1}} + 10\log n.$$

Checking (4) we see that we require

$$\sum_{i=0}^{r} x_i \leq 60p\gamma + 10(\log n)^2$$

to be less than $d_0/2$, and so we need to verify

$$\frac{600np\log n}{\beta} + 10(\log n)^2 \le \frac{\beta}{4}. \tag{7}$$

Note that (5) follows immediately from (7). Since $np > (\log n)^K$, $K > \max\{\frac{2\alpha}{\alpha-1}, \frac{\alpha}{\alpha-2}\}$ and $\alpha > 2$ we have that

$$\beta \ge \Omega\left(\frac{(np)^{\frac{\alpha-1}{\alpha}}}{\log np}\right) \gg \max\left\{(\log n)^2, (np)^{1/\alpha}\log n\right\},$$

which implies (7).

4 Proof of Theorem 1.2

We start by proving part (a) of Theorem 1.2. Suppose that $2/n \le p \le 1-\eta$ and the number of colors is at most $k = \frac{n}{10(\log n)(\log_b np)}$. Also recall that if $p = o(1)$ then $\log_b x = (1+o(1))\frac{\log x}{p}$.

Breaker employs the following strategy. He chooses one part of the bipartite graph which we denote by W_B to be *Breaker's* side and thinks of the opposite part W_M as *Maker's* side. Loosely speaking, Breaker tries to eliminate the coloring possibilities on Maker's side. In order to state Breaker's strategy precisely, we introduce a definition. We say that a color is *dead* if it is available on less than $6(\log n)(\log_b np)$ vertices on Maker's side W_M. Breaker colors according to the following three simple rules:

1. Only color on Breaker's side W_B,
2. Do not use a dead color, and
3. If possible, respond to a move by Maker on Maker's side in kind (i.e. when Maker plays on Maker's side with a particular color then Breaker's first choice is to play the same color on Breaker's side).

We say that a color *escapes* if it is not dead and Breaker stops playing this color because he cannot choose a vertex on his side that can be colored with this color. Note that it follows from the third rule for Breaker that the number of times a color is played on Breaker's side is at least the number times it is played on Maker's side as long as the color is neither dead nor has escaped. Note further that there may be rounds when Breaker's move will not be dictated by the rules above. During these rounds Breaker simply colors arbitrarily on Breaker's side. We continue play until every color either dies or escapes; that is, we play until Breaker cannot follow his coloring rules. Suppose that this happens after $\nu_M \le \nu_B$ vertices have been colored on Maker's and Breaker's sides respectively. We will show that **whp** Breaker will be in a winning position by this time.

Recall that W_M, W_B denote Maker and Breakers' sides of the bipartition. For $X \subseteq W_B$ we let $\overline{N}(X)$ denote the set of vertices in W_M that have no neighbors in X (i.e. $\overline{N}(X) = W_M \setminus N(X)$). Let

$$\lambda_0 = \log_b n - \log_b \log n - \log_b \log_b np - \log_b 3$$

and note that

$$\lambda_0 < \log_b np \qquad \text{and} \qquad n(1-p)^{\lambda_0} = 3(\log n)(\log_b np). \tag{8}$$

Lemma 4.1 Whp *every subset $L \subseteq W_B$ of size $\ell \leq \lambda_0$ has at most $2n(1-p)^\ell$ non-neighbors in W_M.*

Proof. Fix $L \subseteq W_B$ with $|L| = \ell$. The number of non-neighbors of L in W_M is distributed as the binomial $B(n,(1-p)^\ell)$. Thus, by the Chernoff bounds,

$$\mathbb{P}(\exists L : |\overline{N}(L)| \geq 2n(1-p)^\ell) \leq \sum_{\ell=1}^{\lambda_0} \binom{n}{\ell} e^{-n(1-p)^\ell/3} \leq \sum_{\ell=1}^{\lambda_0} \left(\frac{ne}{\ell}\right)^\ell e^{-n(1-p)^\ell/3}.$$

Now if $\ell \leq \lambda_0 < \log_b np$ then

$$\log(n^\ell e^{-n(1-p)^\ell/3}) = \ell \log n - n(1-p)^\ell/3 \leq \begin{cases} -\sqrt{n} & \text{if } \ell \leq \sqrt{\log n} \\ 0 & \text{if } \sqrt{\log n} \leq \ell \leq \lambda_0 \end{cases}.$$

Therefore,

$$\mathbb{P}\big(\exists L : |\overline{N}(L)| \geq 2n(1-p)^\ell\big) \leq \sum_{\ell=1}^{\sqrt{\log n}} \left(\frac{e}{\ell}\right)^\ell e^{-\sqrt{n}} + \sum_{\ell=\sqrt{\log n}}^{\lambda_0} \left(\frac{e}{\ell}\right)^\ell = o(1).$$

□

It follows from (8), Lemma 4.1 and the definition of a dead color that Breaker makes a rule based use of each color at most λ_0 times. Using the fact that at least as many vertices will be colored on Breaker's side as on Maker's side and that they both had the same number of turns we conclude that the number of colored vertices at the point when Breaker stops satisfies

$$\nu_M \leq \nu_B \leq 2k\lambda_0 \leq \frac{n}{5\log n}.$$

Let $c_1, \ldots, c_t$ be the colors that escape. Let M_i, B_i be the sets of vertices with color c_i on Maker's and Breaker's sides respectively at the moment that Breaker stops playing color c_i because he is forced to by the rules. Let $m_i = |M_i|$ and set

$$\alpha = \sum_{i=1}^{t} (1-p)^{m_i}.$$

Note first that

$$m_i \leq b_i = |B_i| \leq \lambda_0, \qquad i = 1, 2, \ldots, t.$$

Furthermore, because $b_i \leq \lambda_0$ we see that

$$|\overline{N}(B_i)| \leq 2n(1-p)^{b_i} \leq 2n(1-p)^{m_i}.$$

We consider two cases.

Case 1. $\alpha < 1/6$.

The total number of vertices that can be colored on Maker's side is at most the sum of (i) the number of vertices colored so far, (ii) the number of vertices that can be colored with dead colors, and (iii) the number of vertices that can be colored with escaped colors. Hence the number of vertices that can be colored on Maker's side is at most

$$\nu_M + k \cdot 6(\log n)(\log_b np) + 2n\sum_{i=1}^{t}(1-p)^{m_i} \leq o(n) + \frac{n}{10(\log n)(\log_b np)} \cdot 6(\log n)(\log_b np) + \frac{n}{3} < n,$$

and therefore Maker can not complete the coloring of the graph.

Case 2. $\alpha \geq 1/6$.

In this case **whp** we arrive at a contradiction. Let Z be the set of ν_B vertices in W_B that have been colored so far. We have $|Z| = \nu_B \leq n/(5\log n)$. When a color c_i escapes it is unavailable to vertices on Breaker's side. It follows that all vertices in $Y = W_B \setminus Z$ have at least one neighbor in M_i.

Let $\mathcal{E}$ be the event that we have such a configuration i.e. t small sets, whose neighborhoods each covers almost all of W_B. Fix the sets $M_1, \ldots, M_t$ and Y. Since $|Y| = (1-o(1))n$, the probability that this collection of sets satisfies the condition is

$$\left(\prod_{i=1}^{t}(1-(1-p)^{m_i})\right)^{|Y|} \leq \exp\left\{-|Y|\sum_{i=1}^{t}(1-p)^{m_i}\right\} \leq e^{-n/7}.$$

The probability of the existence of any such configuration in our random model is at most

$$\sum_{t=1}^{k}\left(\sum_{\ell=0}^{\lambda_0}\binom{n}{\ell}\right)^t \binom{n}{n/(5\log n)} e^{-n/7} \leq n^{\lambda_0 k} e^{o(n)} e^{-n/7} \leq e^{n/10+o(n)-n/7} = o(1)$$

□

4.1 Upper bound in Theorem 1.2

The proof here is essentially the same as for Theorem 1.1(a). Let the vertex bipartition be denoted V_1, V_2. Aside from modifying the statement of Lemma 3.1 to say that the sets in $\mathcal{C}$ are contained in V_i and $v \in V_{3-i}$, the proof goes through basically unchanged.

5 Concluding remarks and open problems

In this paper we obtain upper and lower bounds on the game chromatic number of $G_{n,p}$ which differ only by a multiplicative constant. It would be very interesting to improve our result and determine the asymptotic value of this parameter for random graphs. Our results suggest that in fact the following should be true.

Conjecture 5.1 *If $p \leq 1-\eta$ for some constant $\eta > 0$ and $np \to \infty$, then* **whp**

$$\chi_g(G_{n,p}) = (1+o(1))\frac{n}{\log_b np},$$

where $b = 1/(1-p)$.

We conjecture that the game chromatic number of random bipartite graph $B_{n,p}$ has the same order of magnitude **whp**. We did not succeed in proving the correct lower bound.

As one final remark, there has been much work done on the concentration of the the chromatic number of $G_{n,p}$. None of this is applicable to $\chi_g(G_{n,p})$. Now $\chi_g(G_{n,p})$ should also be concentrated, but proving this may require some new approaches to proving concentration.

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