1. Use the Local Lemma to show
\[ R(3, t) = \Omega \left( \frac{t^2}{\log^2 t} \right). \]

2. Let \( Y_1, Y_2, \ldots, Y_s \) be chosen uniformly and independently at random from \([m] := \{1, 2, \ldots, m\}\) and set \( Y = \{Y_1, \ldots, Y_s\} \).
   (a) Show that for any \( \emptyset \neq A \subseteq [m] \) and \( i \in [m] \setminus A \) we have
   \[ Pr(i \in Y \mid A \subseteq Y) \leq Pr(i \in Y). \]
   (b) Show
   \[ Pr(Y = [m]) < [1 - (1 - 1/m)^s]^m. \]
   
   Hint: Use a coupling.

For a graph \( G = (V, E) \) and sets of colors \((S(v) : v \in V)\), with each \( S(v) \) a subset of some universal set of colors \( \Gamma \), a coloring \( \sigma : V \to \Gamma \) is \( S \)-legal if it is a proper coloring (i.e. adjacent vertices get different colors) and \( \sigma(v) \in S(v) \) for all \( v \in V \).

The list-chromatic number of \( G \), denoted \( \chi_l(G) \), is the smallest \( t \) such that for every choice of \( \{S(v) : v \in V\} \) such that \( |S(v)| = t \ \forall v \) there exists an \( S \)-legal coloring.

3. Show that for a bipartite graph \( G \) of maximum degree \( \Delta \) we have
\[ \chi_l(G) = O(\Delta/(\log \Delta)). \]

4. Let \( k \geq 1 \) be fixed. Let \( G = (V, E) \) be a simple graph, and let \( S(v) \) be a set of at least \( 10k \) colors for each \( v \in V \). Assume that for each \( v \in V \) and each color \( \gamma \) we have
\[ |\{w : w \sim v \text{ and } \gamma \in S(w)\}| \leq k. \]

Prove that \( G \) has an \( S \)-legal coloring.

5. We recall some material from homework 1.

Let \( G \) be a triangle-free graph on \( n \) vertices with maximum degree \( \Delta \), and let \( \mathcal{I} \) be the collection of all independent sets in \( G \). We defined the independence polynomial of \( G \) as
\[ P_G(x) = \sum_{I \in \mathcal{I}} x^{|I|}, \]
and we considered the probability distribution on \( \mathcal{I} \) given by
\[ Pr(I) = Pr_\gamma(I) = \frac{\gamma^{|I|}}{P_G(\gamma)}. \]
where $\gamma > 0$. We further defined the random variable $X$ to be the cardinality of an independent set chosen at random according to this distribution. Recall that we proved in Homework 1 that we have
\[
\frac{E[X]}{n} \geq \frac{\gamma}{1 + \gamma} \cdot \frac{W(\Delta \log(\gamma + 1))}{\Delta \log(\gamma + 1)}.
\]
where $W(z)$ denotes the unique positive real such that $W(z)e^{W(z)} = z$.

Prove that $E[X]$ is monotone increasing as function of $\lambda$, and conclude that we have
\[
\frac{1}{|I|} \sum_{I \in I} |I| \geq (1 + o(1)) \frac{n \log(\Delta)}{\Delta}.
\]