1. Let $G$ be a 2-connected planar graph with $n$ vertices and degree sequence $(d_i)_{i=1}^n$. Show that
\[ \sum_{i:d_i \leq 6} (6 - d_i) \geq \sum_{i=1}^n (6 - d_i) \geq 12. \]
Deduce that if $\delta(G) \geq 5$ then $G$ has at least 12 vertices of degree 5, and if $\delta(G) \geq 4$ then $G$ has at least 6 vertices of degree at most 5.

2. A graph is **outerplanar** if it can be embedded in the plane so that every vertex is on the boundary of a single face.
   
   (a) Use Euler’s formula to prove that $K_4$ and $K_{2,3}$ are not outerplanar.
   
   (b) Prove that a graph $G$ is outerplanar if and only if $G$ contains neither $K_4$ nor $K_{2,3}$ as a minor.

3. Show that a 2-connected plane graph $G$ is bipartite if and only if the boundary of every face of $G$ is an even cycle.

Let $G = (V, E)$ be a 2-connected plane graph, and assume that the intersection of the boundaries of any pair of faces contains at most one edge. Let $\mathcal{F}$ be set of faces $G$. We define the dual of $G$ to be the graph $G^*$ which has vertex set $\mathcal{F}$ and an edge joining $F_1, F_2 \in \mathcal{F}$ if and only if the boundaries of $F_1$ and $F_2$ meet in an edge. For each edge $e \in E$ let $e^*$ be the edge in $G^*$ that joins the two faces that have $e$ on their boundary. Note that the map from $E$ to $E(G^*)$ that takes $e$ to $e^*$ is a natural bijection between $E$ and $E(G^*)$.

4. Let $G$ be a 2-connected plane graph, and assume that the intersection of the boundaries of any pair of faces contains at most one edge. Show that $\chi(G^*) = 2$ if and only if the degree of every vertex in $G$ is even.
   
   **Hint:** Recall the definition of an Eulerian graph, and find a decomposition of the $G$ into edge-disjoint cycles.

5. Let $G = (V, E)$ be a 2-connected plane graph, and assume that the intersection of the boundaries of any pair of faces contains at most one edge. Let $T$ be a spanning tree of $G$. Show that the graph on vertex set $\mathcal{F}$ with edge set
\[ \{e^*: e \in E \setminus E(T)\} \]

is a spanning tree of $G^*$.