1. Show that a graph $G = (V, E)$ has a matching of size $k$ if and only if

$$q(G - S) \leq |S| + |V| - 2k$$

for all $S \subseteq V$.

2. Deduce the Marriage Theorem from Tutte’s 1-factor Theorem.

3. Prove that every connected graph has a vertex that is not a cut-vertex.

4. Prove that if $G = (V, E)$ is 2-connected and

$$\kappa(G - x) = 1 \quad \text{for all } x \in V$$

then $\delta(G) = 2$.

Hint. Consider a vertex-cut $\{x, y\}$ such that $G - x - y$ has a connected component on the minimum number of vertices.

For the remaining questions make use of the following definitions. If $G$ is a graph on more than one vertex and $G - F$ is connected for every set $F$ of fewer than $\ell$ edges then we say that $G$ is $\ell$-edge-connected. The greatest integer $\ell$ such that $G$ is $\ell$-edge-connected is the edge connectivity of $G$, which is denoted $\lambda(G)$.

5. Let $G$ be an $n$ vertex graph such that

$$d(x) + d(y) \geq n - 1 \quad \text{for all non-adjacent } x, y \in V(G).$$

Prove that $\lambda(G) = \delta(G)$.

6. Suppose $G = (V, E)$ is $k$-edge-connected and the deletion of any edge of $G$ gives a graph that is not $k$-edge-connected. Show that $G$ has minimum degree $k$.

Hint: Consider $X \subseteq V$ with $|E(X, V \setminus X)| = k$ and $|X|$ minimum.