21-228 Discrete Mathematics
Course Review 1

This document contains a list of the important definitions and theorems that have been covered thus far in the course. It is *not* a complete listing of what has happened in the course. The sections from the book that correspond with each topic are also given. Following the list of important definitions and theorems you will find a collection of review exercises.

The Sections listed in italics are from *Invitation to Discrete Mathematics* by Matoušek and Nešetřil.

1. Counting sets and functions. *Sections 2.1 - 2.3.*

   Note 1. *We use the notation conventions* $\mathbb{N} = \{0, 1, 2, \dots\}$ *and* $\mathbb{N}^+ = \{x \in \mathbb{N} : x > 0\} = \{1, 2, \dots\}$.

   Definition 2. *Let S be a set. A* **partition** *of S is a collection of sets $A_1, \dots, A_k$ such that*

   $$S = \bigcup_{i=1}^{k} A_i, \quad \text{and}$$
   $$A_i \cap A_j = \emptyset \quad \text{for all } i \neq j.$$

   Recall: The equivalence classes of an equivalence relation on S form a partition of S.

   Note 3. *If S is a finite set and $A_1, \dots, A_k$ form a partition of S then*

   $$|S| = \sum_{i=1}^{k} |A_i|.$$

   Definition 4. *If X is a finite set then 2^X is the collection of all subsets of X; to be precise,*

   $$2^X = \{A : A \subseteq X\}.$$

   Claim 5. *If X is a finite set then $|2^X| = 2^{|X|}$.*

   Definition 6. *Let $X = \{1, \dots, n\}$. The* **characteristic vector** *of a set $A \subseteq X$ is $1_A = (y_1, \dots, y_n)$ where*

   $$y_i = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{if } i \notin A. \end{cases}$$

   Definition 7. *Let X be a finite set. For $k \in \mathbb{N}$ we define $\binom{X}{k}$ to be the collection of all k-element subsets of X; that is,*

   $$\binom{X}{k} = \{A \subseteq X : |A| = k\}.$$

   Claim 8. *If X is a finite set and $k \in \mathbb{N}$ then*

   $$\left|\binom{X}{k}\right| = \frac{n!}{k!(n-k)!} =: \binom{n}{k}.$$

Claim 9. *If $n \in \mathbb{N}$ then*

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Claim 10. *If $0 \le k \le n$ then*

$$\binom{n}{k} = \binom{n}{n-k}.$$

Claim 11. *If $0 \le k \le n$ then*

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Definition 12. *Let X and Y be sets. X^Y is the collection of functions from Y to X. There is a correspondence between the functions in X^Y and the strings of elements of X indexed by Y. If $Y = \{1, \ldots, s\}$ then the string $(x_1, \ldots, x_s)$ corresponds to the function f such that $f(i) = x_i$ for $i = 1, \ldots, s$.*

Definition 13. *For $m, k \in \mathbb{N}^+$, $S(m, k)$ is defined to be the number of partitions of a set of size m into k nonempty parts. These are called the* **Stirling numbers of the second kind.**

Claim 14. *Let X and Y be finite sets such that $|X| = n$ and $|Y| = s$. We have*

$$\begin{aligned} |X^Y| &= |X|^{|Y|} = n^s \\ \left|\left\{f \in X^Y : f \text{ is an injection}\right\}\right| &= n(n-1)\cdots(n-s+1) \\ \left|\left\{f \in X^Y : f \text{ is a surjection}\right\}\right| &= S(s,n)n!. \end{aligned}$$

Note that if $n < s$ then the number of injections is 0. Furthermore, if $s < n$ then the number of surjections is 0.

Claim 15. *Let Z be a collection of k* **indistinguishable** *objects and let X be a set of n* **distinguishable** *labels. We consider labellings of the objects of Z with the elements of X. We consider two such labellings distinct if there is a label in X that appears a different number of times in the two labellings. So, there is correspondence between the set of distinct labeling and the set*

$$\{(x_1, \ldots, x_n) : x_1, \ldots, x_n \in \mathbb{N} \quad \text{and} \quad x_1 + \cdots + x_n = k\}.$$

(In this correspondence x_i is the number of times label i is used.) We have

$$\text{total number of labellings} \quad = \quad \tbinom{n+k-1}{n-1}$$

2. APPROXIMATIONS. *Sections 2.4 - 2.6.*

Definition 16. *If $f(n)$ and $g(n)$ are functions from $\mathbb{N}$ to $\mathbb{R}$ then write $f \sim g$ for*

$$\lim_{n\to\infty} f/g = 1.$$

Theorem 17 (Stirling's Formula)**.**

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}.$$

Claim 18. *For any $n \in \mathbb{N}^+$ we have*

$$e\left(\frac{n}{e}\right)^n \le n! \le \frac{e^2}{4}(n+1)\left(\frac{n}{e}\right)^n.$$

Theorem 19 (Binomial Theorem)**.** *For any $n \in \mathbb{N}^+$ we have*

$$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r.$$

(Note: this holds over any field.)

Definition 20. *For $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}^+$ we set*

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}.$$

For $\alpha \in \mathbb{R}$ we set $\binom{\alpha}{0} = 1$.

Theorem 21 (Newton's Binomial Theorem)**.** *If $\alpha, x \in \mathbb{R}$ and $|x| < 1$ then*

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k.$$

Claim 22. *If $0 \le k < n$ are integers then*

$$\binom{n}{k}(n-k) = \binom{n}{k+1}(k+1)$$

Note 23. *Every row of Pascal's triangle is symmetric and has its maximum in the middle. In particular,*

$$\binom{n}{0} < \binom{n}{1} < \binom{n}{2} < \cdots < \binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil} > \cdots > \binom{n}{n-2} > \binom{n}{n-1} > \binom{n}{n}.$$

Claim 24. *If $0 \le t < m$ are integers then*

$$e^{-\frac{t^2}{m-t+1}} \le \frac{\binom{2m}{m-t}}{\binom{2m}{m}} \le e^{-\frac{t^2}{m+t}}.$$

Lemma 25. *For every $x > 0$*

$$\frac{x-1}{x} \le \log x \le x - 1$$

(Where log *denotes the natural logarithm.)*

Lemma 26. *For all x we have*

$$1 + x \le e^x.$$

3. Inclusion/Exclusion. *Sections 2.7 - 2.8.*

Theorem 27 (Principle of inclusion and exclusion)**.** *Let $A_1, \ldots, A_n$ be subsets of a finite set Ω. For $S \subseteq \{1, \ldots, n\}$ set*

$$A_S = \begin{cases} \cap_{i \in S} A_i & \text{if } S \neq \emptyset \\ \Omega & \text{if } S = \emptyset. \end{cases}$$

We have

$$\left| \Omega \setminus \left(\bigcup_{i=1}^{n} A_i \right) \right| = \sum_{S \subseteq \{1, \ldots, n\}} (-1)^{|S|} |A_S|$$

Lemma 28. *If $n \ge 1$ is an integer then*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Claim 29. *Let X be a finite set. A bijection $\sigma : X \to X$ is a* **derangement** *if $\sigma(x) \neq x$ for all $x \in X$. The number of derangements of a set of cardinality n is*

$$n! \sum_{k=0}^{n} (-1)^k \frac{1}{k!}.$$

Claim 30.

$$S(n, k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k - i)^n.$$

Theorem 31 (Bonferroni Inequalities)**.** *Let $A_1, \ldots, A_n$ be subsets of a finite set Ω. For $S \subseteq \{1, \ldots, n\}$ set*

$$A_S = \begin{cases} \cap_{i \in S} A_i & \text{if } S \neq \emptyset \\ \Omega & \text{if } S = \emptyset. \end{cases}$$

If k is even then we have

$$\left| \left(\bigcup_{i=1}^{n} A_i \right) \right| \ge \sum_{j=1}^{k} \sum_{S \in \binom{\{1,2,\ldots,n\}}{j}} (-1)^{j+1} |A_S|$$

If k is odd then we have

$$\left| \left(\bigcup_{i=1}^{n} A_i \right) \right| \le \sum_{j=1}^{k} \sum_{S \in \binom{\{1,2,\ldots,n\}}{j}} (-1)^{j+1} |A_S|$$

Lemma 32. *Let $1 \le k \le n$ be integers. If k is even then*

$$\sum_{j=1}^{k} (-1)^{j+1} \binom{n}{j} \le 1.$$

If k is odd then

$$\sum_{j=1}^{k}(-1)^{j+1}\binom{n}{j} \geq 1.$$

4. GENERATING FUNCTIONS. *Section 10.1.*

Definition 33. *Let $a_0, a_1, \ldots$ be an infinite sequence. The* **generating function** *for this sequence is*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Note 34. *A generating function can be viewed as either*

(i) A function of x (when we have convergence).

(ii) A formal object with addition and multiplication.

Notation 35. *If $f(x)$ is a polynomial in x or a generating function then we define $[x^n]f(x)$ to be the coefficient of x^n in f. For example, if $f(x) = 1 + 22x^2 + 3x^3 + 5x^4$ then $[x^2]f(x) = 22$.*

Note 36. *If $f(x), g(x)$ are polynomials in x or generating functions then*

$$[x^n]f(x)g(x) = \sum_{i=0}^{n}[x^i]f(x) \cdot [x^{n-i}]g(x).$$

REVIEW EXERCISES: Working the following problems should help in preparation for the test. They are not necessarily 'sample' test questions.

1. Let $D(n)$ denote the number of derangements of an n-element set. Give a **combinatorial** proof of the following identity.

$$n! = D(n) + \binom{n}{1}D(n-1) + \cdots + \binom{n}{i}D(n-i) + \cdots + \binom{n}{n-2}D(2) + \binom{n}{n-1}D(1) + 1.$$

2. How many integral solutions of the the equation

$$x_1 + x_2 + x_3 + x_4 = 18$$

satisfy the inequalities $0 \leq x_1 \leq 5$, $0 \leq x_2 \leq 10$, $0 \leq x_3 \leq 7$ and $0 \leq x_4 \leq 13$?

Hint: Begin with the set of integral solutions such that each x_i is non-negative.

3. Compute the sum

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}\binom{n}{n-i}$$

4. Let $n \in \mathbb{N}^+$. Show that

$$2^{n/2}\cos\left(\frac{n\pi}{4}\right) = \sum_{0 \leq k \leq n : 2|k} (-1)^{k/2}\binom{n}{k} = \binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \cdots$$

Hint: Expand $(1+i)^n$ using the binomial theorem (where $i^2 = -1$). Use the identities $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ and $1 + i = \sqrt{2}(cos(\pi/4) + i\sin(\pi/4))$.

5. Use the binomial theorem to prove the following:

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.$$

6. Let T be a set such that $|T| = n$. How many sequences $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k$ such that $S_i \subseteq T$ for $i = 1, \ldots k$ are there?

7. Determine the number of permutations of $\{1, 2, \ldots, 8\}$ in which no even integer is in its natural position.

8. How many positive integers $n < 100$ are not divisible by a square of any integers greater than 1?

9. Prove the inequality

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{n} \geq \log(n+1).$$

10. Let $\epsilon > 0$ be a constant. There exists a constant $\alpha > 0$ such that $k < \alpha n$ implies

$$\binom{n}{k} < (1+\epsilon)^n$$

for all n.

11. Let X and Y be sets such that $|X| = 3n$ and $|Y| = n$. Use inclusion/exclusion to write a formula for the number of surjective functions from X to Y. (Your formula may include the Stirling numbers of the second kind.)

12. A bakery sells croissants, chocolate chip cookies, and oatmeal cookies. How many ways are there to buy 15 items so that you buy at least 2 of each of the two types of cookies and at least 3 croissants. Write your answer as the coefficient of a polynomial.

From Lovász, Pelikán, Vesztergombi: 1.8.26, 1.8.31, 1.8.33, 1.8.34, 2.1.13, 2.3.1, 2.5.5, 2.5.6, 3.2.3, 3.4.2, 3.7.2, 3.8.6, 3.8.8, 3.8.10, 3.8.12, 3.8.13, 3.8.14.

Sample Questions: The test will include some questions in the following format:

For each of the following statements, say whether the statement is true or false and give a short justification for your answer.

1. $\binom{n}{10} \sim n^{10}/(10!)$.
2. If $n, k \in \mathbb{N}^+$ and $0 \leq k \leq n$ then
$$\binom{n}{k} \leq 2^n.$$
3. More than half of the numbers in the set $\{0, 1, 2, \ldots, 10^{10} - 1\}$ contain the digit 9 (in their decimal expansion).
4. There exists $n \in \mathbb{N}^+$ such that
$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \geq 2\sqrt{n}$$
5. If X is a finite set then
$$\sum_{S \subseteq X} (-1)^{|S|} = 0.$$
6. If $n \geq 3$ and $1 < k < n$ then $S(n, k) = S(n-1, k-1) + kS(n-1, k)$.