1. A collection of \( n \) teams plays a round robin tournament. (I.e. a tournament in which every team plays every other team exactly once. There will be \( \binom{n}{2} \) games in this tournament). We say that the tournament has property \( S_k \) if for every set \( S \) of \( k \) teams there is some team that beats all teams in the set \( S \). For example if we have 3 teams then the tournament would have property \( S_1 \) if, for example, team 1 beats team 2, team 2 beats team 3 and team 3 beats team 1. Prove that if
\[
\binom{n}{k} (1 - 2^{-k})^{n-k} < 1
\]
then it is possible that a tournament with \( n \) teams has property \( S_k \).

2. Which of the following statements are true for any random variable \( X \) defined on a finite probability space with the uniform distribution? Provide proofs for the true statements and counterexamples for the false statements.

   (a) \( E[1/X] = 1/E[X] \).

   (b) \( E[X]^2 \leq E[X^2] \).

   (c) \( \sqrt{E[X]} \geq E[\sqrt{X}] \).

3. Let \( p_2, p_3, \ldots \) be a sequence of real numbers such that \( 0 < p_n < 1 \) for all \( n \), and consider the following sequence of probability spaces. The set \( \Omega_n \) consists of all graphs on vertex set \( V_n \) where \( |V_n| = n \), and for every graph \( G = (V_n, E) \) we set
\[
P(G) = p^{\binom{|E|}{2}} (1 - p)^{\binom{n}{2} - |E|}.
\]
Note that this probability function is given by using \( \binom{n}{2} \) mutually independent biased coin flips to determine whether or not each element of \( \binom{V_n}{2} \) appears as an edge. (This object is called the binomial random graph or the Erdős-Rényi random graph.)

   (a) Let the random variable \( X_n(G) \) count the number of copies of \( K_3 \) in the graph \( G \). Determine \( E[X_n] \).

   (b) Suppose \( p_n = 1/n \). Prove \( P(X_n = 0) > 1/2 \) for all \( n \).

4. Consider the binomial random graph (as defined in the previous problem) with \( p_n = \frac{\log(n)}{n} \), and again let the random variable \( X_n \) count the number of copies of \( K_3 \). Use Chebyshev’s inequality to prove
\[
\lim_{n \to \infty} P(X_n = 0) = 0.
\]

   Hint. Use indicator random variables for both the expectation and the variance.
5. Prove that if $G = (V, E)$ is a graph such that $|V| = n$ and

$$|E| \geq \frac{(n - 1)(n - 2)}{2} + 1$$

then $G$ is connected. Give an example of a disconnected graph on $n$ vertices with $(n - 1)(n - 2)/2$ edges.

6. If $C$ is a cycle and $e$ is an edge connecting two non-consecutive vertices in $C$ then we call $e$ a chord of $C$.

(a) Prove that if every vertex in a graph $G$ has degree at least 2 then $G$ contains a cycle.

(b) Prove that if every vertex of a graph $G$ has degree at least 3 then $G$ contains a cycle with a chord.

*Hint: Consider a maximal path.*