1. Suppose 6n fair dice are rolled. Define a probability space that describes this experiment. Let $p_n$ be the probability that every number (i.e. the numbers 1 through 6) appears exactly $n$ times in one roll of the collection of dice. Give an exact expression for $p_n$. Find a simpler function $f_n$ such that $p_n \sim f_n$.

2. Prove that if there is a number $p$ such that $0 \leq p \leq 1$ and

$$\binom{n}{k} p^k + \binom{n}{t} (1 - p)^t < 1,$$

then the Ramsey number $R(k, t)$ satisfies $R(k, t) > n$.

3. Suppose we color the edge set of $K_n$ with 2 colors uniformly at random (so, we have $|\Omega| = 2^{\binom{n}{2}}$ and the uniform distribution).

(a) Let $X$ be the number of monochromatic triangles in the random coloring. Determine $E(X)$.

(b) Conclude that there exists a coloring of the edge set of $K_n$ that has at most $\frac{1}{4} \binom{n}{3}$ monochromatic triangles.

4. A collection of $k$ people enter an elevator in a building that has $n + 1$ floors. Each of the $k$ people has a destination floor that is chosen uniformly at random from floors $2, \ldots, n + 1$. What is the expected number of stops that the elevator makes?

5. Consider the probability space on the finite set $\Omega$ with the uniform distribution. Let $X$ be a random variable that takes values in the set $\{0, 1, 2, \ldots, M\}$ such that $E[X] = M - a$. Prove that for any $1 \leq b \leq M$ we have

$$\mathbb{P}(X \geq M - b) \geq \frac{b - a}{b}.$$

6. If $f$ is a permutation of the set $X = \{1, 2, \ldots, n\}$ then a cycle in $f$ is a sequence of distinct elements $a_1, a_2, \ldots, a_k$ of $X$ with the property that $f(a_i) = a_{i+1}$ for $i = 1, \ldots, k - 1$ and $f(a_k) = a_1$. The number of elements in the sequence is the length of the cycle. Note that every element of $X$ is in a unique cycle of $f$ and that an element $x \in X$ such that $f(x) = x$ is a cycle of length 1.

Let $\Omega$ be the set of permutations of the set $\{1, 2, \ldots, n\}$. Consider the probability space on $\Omega$ given by the uniform distribution. Determine the expected length of the cycle containing 1 in a permutation chosen at random from this probability space.