1. Find all positive integers $a > b > c$ for which

$$\binom{a}{b} \binom{b}{c} = 2 \binom{a}{c}.$$ 

Give a combinatorial justification for your answer (i.e. appeal to the fact that $\binom{a}{b}$ gives the number of subsets of an $a$-element set having exactly $b$ elements).

2. If $m$ indistinguishable 4-sided die are rolled how many distinguishable outcomes are there? (E.g. if $m = 2$ then there are 10 possibilities.) How many outcomes are there in which each of the four numbers appear at least once?

3. Let $p$ and $q$ be positive integers. How many sequences of $p$ 1’s and $q$ 0’s are there with the property that there are at least two 0’s between every pair of 1’s?

4. A function $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is monotone if $i < j$ implies $f(i) \leq f(j)$. How many monotone functions $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ are there?

5. Let $F$ be a collection of $k$-element subsets of the set $X$, where $|X| = n$. The shadow of $F$ is defined to be the set

$$\partial F = \left\{ B \in \binom{X}{k-1} : \exists A \in F \text{ such that } B \subset A \right\}.$$ 

Show

$$|\partial(F)| \geq \frac{k|F|}{n-k+1}.$$ 

Hint. Count ordered pairs $(A, B)$ such that $|A| = k$, $|B| = k-1$, $B \subset A$ and $A \in F$.

6. Use a combinatorial argument to prove the following: for all positive integers $m_1, m_2, n$ we have

$$\sum_{k=0}^{n} \binom{m_1}{k} \binom{m_2}{n-k} = \binom{m_1 + m_2}{n}.$$