1 Modica–Mortola Functional

2 Γ-Convergence

Let \((X, d)\) be a metric space and consider a sequence \(\{F_n\}\) of functionals \(F_n : X \to [-\infty, \infty]\). We say that \(\{F_n\}\) Γ-converges to a functional \(F : X \to [-\infty, \infty]\) if the following properties hold:

(i) (Liminf Inequality) For every \(x \in X\) and every sequence \(\{x_n\} \subset X\) such that \(x_n \to x\),

\[
F(x) \leq \liminf_{n \to +\infty} F_n(x_n). \tag{1}
\]

(ii) (Limsup Inequality) For every \(x \in X\) there exists \(\{x_n\} \subset X\) such that \(x_n \to x\) and

\[
\limsup_{n \to +\infty} F_n(x_n) \leq F(x). \tag{2}
\]

**Remark 1** An important property of Γ-convergence is that if each \(F_n\) is bounded from below and admits a minimizer \(x_n\), that is,

\[
F_n(x_n) = \min_{y \in X} F_n(y),
\]

if \(x_n \to x\) for some \(x \in X\), and if \(\{F_n\}\) Γ-converges to \(F\), then \(x\) is a minimizer of \(F\) and

\[
\min_{y \in X} F(y) = F(x) = \lim_{n \to +\infty} F_n(x_n) = \lim_{n \to +\infty} \min_{y \in X} F_n(y).
\]

To see this, let \(y \in Y\) and apply property (ii) to find a sequence \(\{y_n\} \subset X\) such that \(y_n \to y\) and

\[
\limsup_{n \to +\infty} F_n(y_n) \leq F(y).
\]

Using the fact that \(F_n(x_n) = \min_{y \in X} F_n(y)\) and property (i) for the sequence \(\{x_n\}\), we have that

\[
F(x) \leq \liminf_{n \to +\infty} F_n(x_n) \leq \limsup_{n \to +\infty} F_n(x_n) \leq \limsup_{n \to +\infty} F_n(y_n) \leq F(y),
\]

which shows that \(x\) is a minimizer of \(F\). Taking \(y = x\), gives that there exists \(\lim_{n \to +\infty} F_n(x_n) = F(x)\).

3 Compactness

For \(\varepsilon > 0\) consider the functional

\[
F_\varepsilon : W^{1,2}(\Omega; \mathbb{R}^d) \to [0, \infty]
\]
defined by

\[ F_\varepsilon (u) := \int_\Omega \left( \frac{1}{\varepsilon} W(u) + \varepsilon |\nabla u|^2 \right) \, dx, \]

where the double well potential \( W : \mathbb{R}^d \to [0, \infty) \) satisfies the following hypotheses:

(H1) \( W \) is continuous, \( W(z) = 0 \) if and only if \( z \in \{ \alpha, \beta \} \) for some \( \alpha, \beta \in \mathbb{R}^d \) with \( \alpha \neq \beta \).

(H2) There exist \( L > 0 \) and \( R > 0 \) such that

\[ W(z) \geq L |z| \]

for all \( z \in \mathbb{R}^d \) with \( |z| \geq R \).

**Theorem 2** Let \( \Omega \subset \mathbb{R}^N \) be an open bounded set with Lipschitz boundary. Assume that the double-well potential \( W \) satisfies conditions (H1) and (H2). Let \( \varepsilon_n \to 0^+ \) and let \( \{u_n\} \subset W^{1,2} (\Omega; \mathbb{R}^d) \) be such that

\[ M := \sup_n F_{\varepsilon_n} (u_n) < \infty. \]  

Then there exist a subsequence \( \{u_{n_k}\} \) of \( \{u_n\} \) and \( u \in BV (\Omega; \{\alpha, \beta\}) \) such that

\[ u_{n_k} \to u \text{ for } \text{in } L^1 (\Omega; \mathbb{R}^d). \]

**Proof.** We begin by showing that \( \{u_n\} \) is bounded in \( L^1 (\Omega; \mathbb{R}^d) \) and equi-integrable. By (3) and (H2), for every \( t \geq R \),

\[ \int_{\{ |u_n| \geq t \}} |u_n| \, dx \leq \frac{1}{L} \int_{\{ |u_n| \geq t \}} W(u_n(x)) \, dx \leq M \varepsilon_n, \]

which implies that \( \{u_n\} \) is equi-integrable. Moreover, since \( \Omega \) has finite measure,

\[ L \int_{\Omega} |u_n| \, dx = L \int_{\{ |u_n| \geq R \}} |u_n| \, dx + L \int_{\{ |u_n| < R \}} |u_n| \, dx \leq L \int_{\{ |u_n| \geq R \}} W(u_n(x)) \, dx + LR |\Omega| \leq M \varepsilon_n + LR |\Omega|. \]

In view of the Vitali’s convergence theorem, to obtain strong convergence of a subsequence, it suffices to prove convergence in measure or pointwise \( L^N \) a.e. in \( \Omega \). We divide the proof in two steps.

**Step 1:** Assume first that \( d = 1 \). Define

\[ W_1 (z) := \min \{ W(z), 1 \}, \quad z \in \mathbb{R}. \]

Since \( 0 \leq W_1 \leq W \), for every \( n \in \mathbb{N} \), we have

\[ F_{\varepsilon_n} (u_n) \geq \frac{1}{2} \int_{\Omega} \left( \sqrt{W_1(u_n(x)) |\nabla u_n(x)|} \right) \, dx \]

\[ = \int_{\Omega} (|\nabla (\Phi_1 \circ u_n)(x)|) \, dx, \]
where
\[ \Phi_1(t) := \frac{1}{2} \int_0^t \sqrt{W_1(s)} \, ds, \quad t \in \mathbb{R}. \] (4)

Then by (3),
\[ \sup_n \int_\Omega (|D (\Phi_1 \circ u_n)|) \, dx \leq M. \] (5)

Moreover, since \( pW_1 \geq 1 \)
\[ |\Phi_1 (u_n (x))| \leq |u_n (x)| \]
for \( \mathcal{L}^N \) a.e. \( x \in \Omega \) and for all \( n \in \mathbb{N} \). Since \( \{u_n\} \) is bounded in \( L^1 (\Omega) \), it follows that the sequence \( \{\Phi_1 \circ u_n\} \) is bounded in \( L^1 (\Omega) \). By the Rellich–Kondrachov theorem, there exist a subsequence \( \{u_{nk}\} \) and a function \( w \in BV (\Omega) \) such that
\[ w_k := \Phi_1 \circ u_{nk} \to w \text{ in } L_{loc}^1 (\Omega). \]

By taking a further subsequence, if necessary, without loss of generality, we may assume that \( w_k (x) \to w (x) \) and that \( W (u_{nk} (x)) \to 0 \) for \( \mathcal{L}^N \) a.e. \( x \in \Omega \). Since the function \( W_1 (t) > 0 \) for all \( t \neq \alpha, \beta \), it follows from (4) that the function \( \Phi_1 \) is strictly increasing and continuous. Thus, its inverse \( \Phi_1^{-1} \) is continuous and
\[ u_{nk} (x) = \Phi_1^{-1} (w_k (x)) \to \Phi_1^{-1} (w (x)) := u (x) \]
for \( \mathcal{L}^N \) a.e. \( x \in \Omega \). It follows by \( (H_1) \) and the fact that \( W (u_{nk} (x)) \to 0 \) for \( \mathcal{L}^N \) a.e. \( x \in \Omega \), that \( u (x) \in \{\alpha, \beta\} \) for \( \mathcal{L}^N \) a.e. \( x \in \Omega \). In turn, \( w (x) \in \{\Phi_1 (\alpha), \Phi_1 (\beta)\} \) for \( \mathcal{L}^N \) a.e. \( x \in \Omega \), and since \( w \in BV (\Omega) \), we may write
\[ w = \Phi_1 (\alpha) \chi_E + \Phi_1 (\beta) (1 - \chi_E) \] (6)
for a set \( E \subset \Omega \) of finite perimeter. Hence,
\[ u = \alpha \chi_E + \beta (1 - \chi_E) \] (7)
belongs to \( BV (\Omega) \).

**Step 2:** Assume that \( d \geq 2 \) and that \( |\alpha| \neq |\beta| \). For every \( t \geq 0 \) define
\[ V (t) := \min_{|z|=t} W (z). \]

Then \( V \) is upper semicontinuous, \( V (t) > 0 \) for \( t \neq |\alpha|, |\beta| \), \( V (|\alpha|) = V (|\beta|) = 0 \), and \( V (t) \geq Lt \) for \( t \geq R \). For every \( u \in W^{1,2} (\Omega; \mathbb{R}^d) \) define
\[ G_\varepsilon (u) := \int_\Omega \left( \frac{1}{\varepsilon} V(|u|) + \varepsilon |\nabla |u| |^2 \right) \, dx \leq F_\varepsilon (u). \]

Then by (3),
\[ \sup_n G_\varepsilon (u_n) \leq \sup_n F_\varepsilon (u_n; \Omega) < \infty, \]
and so by the compactness in the scalar case \(d = 1\), there exist a subsequence \(\{u_{n_k}\}\) and \(w \in BV(\Omega)\) such that

\[
w_k := \Phi_2 \circ |u_{n_k}| \to w \text{ in } L^1_{\text{loc}}(\Omega),
\]

where

\[
\Phi_2(t) := \frac{1}{2} \int_0^t \sqrt{V_1(s)} \, ds, \quad t \in \mathbb{R}
\]

and

\[
V_1(z) := \min \{V(z), 1\}, \quad z \in \mathbb{R}.
\]

Hence,

\[
|u_{n_k}| \to v := \Phi_2^{-1} \circ w \text{ in } L^1(\Omega).
\]

By taking a further subsequence, if necessary, without loss of generality, we may assume that \(u_{n_k}(x) \to v(x)\) and that \(W(u_{n_k}(x)) \to 0\) for \(L^1\) a.e. \(x \in \Omega\). This implies that \(v \in BV(\Omega; \{\alpha, \beta\})\). Define

\[
u(x) := \begin{cases} 
\alpha & \text{if } v(x) = |\alpha|, \\
\beta & \text{if } v(x) = |\beta|.
\end{cases}
\]

We claim that

\[
u_{n_k} \to u \text{ in } L^1(\Omega; \mathbb{R}^d).
\]

To see this, fix \(x \in \Omega\) such that \(|u_{n_k}(x)| \to v(x)\) and that \(W(u_{n_k}(x)) \to 0\). Then, by \((H_1)\), necessarily, \(u_{n_k}(x) \to u(x)\).

**Step 3:** If \(d \geq 2\) and \(|\alpha| = |\beta|\), let \(e_i\) be a vector of the canonical basis of \(\mathbb{R}^d\) such that \(\alpha \cdot e_i \neq \beta \cdot e_i\). Then \(|\alpha + e_i| \neq |\beta + e_i|\). It suffices to apply the previous step with \(W\) replaced by

\[
\hat{W}(z) := W(z - e_i), \quad z \in \mathbb{R}^d,
\]

and \(u_n\) by \(u_n + e_i\).

4. Gamma Convergence of the Modica–Mortola Functional

In view of the previous theorem, the metric convergence in the definition of \(\Gamma\)-convergence should be \(L^1(\Omega; \mathbb{R}^d)\). Thus, we extend \(F_{\varepsilon}\) to \(L^1(\Omega; \mathbb{R}^d)\) by setting

\[
F_{\varepsilon}(u) := \begin{cases} 
\int_\Omega \left( \frac{1}{2} W(u) + \varepsilon |\nabla u|^2 \right) \, dx & \text{if } u \in W^{1,2}(\Omega; \mathbb{R}^d), \\
\infty & \text{if } u \in L^1(\Omega; \mathbb{R}^d) \setminus W^{1,2}(\Omega; \mathbb{R}^d).
\end{cases}
\]

Let \(\varepsilon_n \to 0^+\). Under appropriate hypotheses on \(W\) and \(\Omega\), we will show that the sequence of functionals \(\{F_{\varepsilon_n}\}\) \(\Gamma\)-converges to the functional

\[
F(u) := \begin{cases} 
c W |Du| (\Omega) & \text{if } u \in BV(\Omega; \{\alpha, \beta\}), \\
\infty & \text{if } u \in L^1(\Omega; \mathbb{R}^d) \setminus BV(\Omega; \{\alpha, \beta\}).
\end{cases}
\]
We begin with the liminf inequality. Consider a sequence \( \{u_n\} \subset L^1(\Omega;\mathbb{R}^d) \) such that \( u_n \to u \) in \( L^1(\Omega;\mathbb{R}^d) \) for some \( u \in L^1(\Omega;\mathbb{R}^d) \). If

\[
\liminf_{n \to +\infty} F_{\varepsilon_n}(u_n) = \infty,
\]

then there is nothing to prove, thus we assume that

\[
\liminf_{n \to +\infty} F_{\varepsilon_n}(u_n) < \infty.
\] (8)

Let \( \{\varepsilon_{nk}\} \) be a subsequence of \( \{\varepsilon_n\} \) such that

\[
\liminf_{n \to +\infty} F_{\varepsilon_n}(u_n) = \lim_{k \to +\infty} F_{\varepsilon_{nk}}(u_{nk}) < \infty.
\]

Then \( F_{\varepsilon_{nk}}(u_{nk}) < \infty \) for all \( k \) sufficiently large. Hence, \( u_{nk} \in W^{1,2}(\Omega;\mathbb{R}^d) \) for all \( k \) sufficiently large. Moreover, if hypotheses \((H_1)\) and \((H_2)\), then by Theorem 2, \( u \in BV(\Omega;\{\alpha, \beta\}) \). Finally, by extracting a further subsequence, not relabelled, we can assume that \( u_n(x) \to u(x) \) for \( L^N \) a.e. \( x \in \Omega \).

Hence, in what follow, without loss of generality, we will assume that (8) holds, that \( \{u_n\} \subset W^{1,2}(\Omega;\mathbb{R}^d) \), that \( u \in BV(\Omega;\{\alpha, \beta\}) \), that \( \liminf_{n \to +\infty} F_{\varepsilon_n}(u_n) \) is actually a limit, and that \( \{u_n\} \) converges to \( u \) in \( L^1(\Omega;\mathbb{R}^d) \) and pointwise \( L^N \) a.e. in \( \Omega \).

5 \textbf{Liminf Inequality, } N = 1, \; d = 1

We begin with the case \( N = d = 1 \) and assume that \( (H_1) \) and \( (H_2) \) hold with \( \alpha < \beta \) and that \( \Omega = (a,b) \). Consider \( u \in BV(\Omega;\{\alpha, \beta\}) \). Without loss of generality we may assume that there exists a partition

\[
a = t_0 < t_1 < \cdots < t_m = b
\]

such that \( u(x) = \alpha \) in \( (t_{2i-2}, t_{2i-1}) \) and \( u(x) = \beta \) in \( (t_{2i-1}, t_{2i}) \). Let \( \varepsilon_n \to 0^+ \) and let \( \{u_n\} \subset W^{1,2}(\Omega) \) be such that \( u_n \to u \) in \( L^1(\Omega) \) and pointwise \( L^1 \) a.e. in \( \Omega \). Fix \( \delta > 0 \) small. Then

\[
\int_{\Omega} \left( \frac{1}{\varepsilon_n} W(u_n) + \varepsilon_n |u_n'|^2 \right) dx \geq \sum_{i=1}^{m} \int_{t_i-\delta}^{t_i+\delta} \left( \frac{1}{\varepsilon_n} W(u_n) + \varepsilon_n |u_n'|^2 \right) dx.
\]

Consider one term

\[
\int_{t_i-\delta}^{t_i+\delta} \left( \frac{1}{\varepsilon_n} W(u_n) + \varepsilon_n |u_n'|^2 \right) dx.
\]

For simplicity we can assume that \( t_i = 0 \) and, by taking \( \delta \) smaller, that \( u_n(-\delta) \to \alpha \) and \( u_n(\delta) \to \beta \). Then

\[
\int_{-\delta}^{\delta} \left( \frac{1}{\varepsilon_n} W(u_n(x)) + \varepsilon_n |u_n'(x)|^2 \right) dx = \int_{-\delta/\varepsilon_n}^{\delta/\varepsilon_n} (W(u_n(\varepsilon_n y)) + \varepsilon_n^2 |u_n'(\varepsilon_n y)|^2) dy \]

(9)

\[
= \int_{-\delta/\varepsilon_n}^{\delta/\varepsilon_n} (W(v_n(y)) + |v_n'(y)|^2) dy,
\]
where we have made the change of variables $x = \varepsilon_n y$ and $v_n (y) := u_n (\varepsilon_n y)$. It follows that

\[
\int_{-\delta}^{\delta} \left( \frac{1}{\varepsilon_n} W(u_n (x)) + \varepsilon_n |u'_n (x)|^2 \right) \, dx \geq \int_{-\delta/\varepsilon_n}^{\delta/\varepsilon_n} \left( W(v_n (y)) + (v'_n (y))^2 \right) \, dy \\
\geq \int_{-\delta/\varepsilon_n}^{\delta/\varepsilon_n} 2\sqrt{W(v_n)} v'_n (y) \, dy \\
= \int_{v_n(-\delta)}^{v_n(\delta)} 2\sqrt{W(s)} \, ds \to \int_{\alpha}^{\beta} 2\sqrt{W(s)} \, ds
\]

as $n \to \infty$, where we have made the change of variables $s = w_n (y)$. In turn,

\[
\liminf_{n \to \infty} \int_{\Omega} \left( \frac{1}{\varepsilon_n} W(u_n) + \varepsilon_n |u'_n|^2 \right) \, dx \geq \int_{\alpha}^{\beta} 2\sqrt{W(s)} \, ds \quad \text{(number of jumps of } u) \]

6 Limsup Inequality, $N = 1$, $d = 1$

Assume in addition to $(H_1)$ and $(H_2)$ that $W$ is of class $C^1$. Let $g$ be the solution of the Cauchy problem

\[
g' = \sqrt{W(g)}, \quad g(0) = \frac{\alpha + \beta}{2}.
\]

Then $g$ is globally defined, strictly increasing, $\alpha < g(t) < \beta$ for all $t$, and

\[
\lim_{t \to -\infty} g(t) = \alpha \quad \lim_{t \to \infty} g(t) = \beta.
\]

Define

\[
c_W := \int_{\mathbb{R}} \left( W(g(t)) + |g'(t)|^2 \right) \, dt.
\]

(11)

Note that by (10) we have

\[
c_W = \lim_{n \to \infty} \int_{-n}^{n} \left( W(g(t)) + |g'(t)|^2 \right) \, dt = 2\lim_{n \to \infty} \int_{-n}^{n} \sqrt{W(g(t))} g'(t) \, dt \\
= 2 \lim_{n \to \infty} \int_{v_n(-n)}^{v_n(n)} \sqrt{W(s)} \, ds = 2 \int_{\alpha}^{\beta} \sqrt{W(s)} \, ds.
\]

Hence, if $\Omega = \mathbb{R}$ and

\[
u (y) = \left\{ \begin{array}{ll}
\alpha & \text{if } y < 0, \\
\beta & \text{if } y \geq 0,
\end{array} \right.
\]

then the right sequence would be

\[
u_n (x) := g \left( \frac{x}{\varepsilon_n} \right),
\]

6
\[
\int_{\mathbb{R}} \left( \frac{1}{\varepsilon_n} W(u_n(x)) + \varepsilon_n |u_n'(x)|^2 \right) \, dx = \int_{\mathbb{R}} \left( \frac{1}{\varepsilon_n} W(g \left( \frac{x}{\varepsilon_n} \right)) + \varepsilon_n \frac{1}{\varepsilon_n} g' \left( \frac{x}{\varepsilon_n} \right) u_n'(x) \right)^2 \, dx \\
= \int_{\mathbb{R}} \left( W(g(y)) + |g'(y)|^2 \right) \, dy = c_W.
\]

In the case of a general \( u \in BV(\Omega; \{\alpha, \beta\}) \), we need to glue \( g \) to \( \alpha \) and \( \beta \). Let \( u \) be as in the previous section. Fix \( \rho > 0 \) and let \( b_\rho \) be such that \( g(b_\rho) = \beta - \rho \) and let \( a_\rho \) be such that \( g(a_\rho) = \alpha + \rho \). Define
\[
g_\rho (y) := \begin{cases} 
\beta & \text{if } y \geq b_\rho + 1, \\
\rho(y - b_\rho) + \beta - \rho & \text{if } b_\rho < y < b_\rho + 1, \\
g(y) & \text{if } a_\rho \leq y \leq b_\rho, \\
\rho(y - a_\rho) + \alpha + \rho & \text{if } a_\rho - 1 < y < a_\rho, \\
\alpha & \text{if } y \leq a_\rho - 1,
\end{cases}
\]
and
\[
u_{n, \rho}(x) := \begin{cases} 
g_\rho \left( \frac{x - t_{2i}}{\varepsilon_n} \right) & \text{if } t_{2i} - r < x < t_{2i} + r, \\
g_\rho \left( \frac{t_{2i+1} - x}{\varepsilon_n} \right) & \text{if } t_{2i+1} - r < x < t_{2i+1} + r, \\
u(x) & \text{otherwise},
\end{cases}
\]
where \( 2r < \min \{t_{i+1} - t_i : i = 0, \ldots, m - 1\} \). Then using the change of variables \( \frac{x - t_{2i}}{\varepsilon_n} = y \) or \( \frac{t_{2i+1} - x}{\varepsilon_n} = y \) we get
\[
\int_{\Omega} \left( \frac{1}{\varepsilon_n} W(u_{n, \rho}) + \varepsilon_n |u_{n, \rho}'|^2 \right) \, dx = \sum_{i=1}^{m} \int_{t_{i} - r}^{t_{i} + r} \left( \frac{1}{\varepsilon_n} W(u_{n, \rho}) + \varepsilon_n |u_{n, \rho}'|^2 \right) \, dx \\
= \sum_{i=1}^{m} \int_{-r/a_n}^{r/a_n} \left( W(g_\rho(y)) + |g_\rho'(y)|^2 \right) \, dy \\
= \sum_{i=1}^{m} \int_{a_\rho - 1}^{b_\rho + 1} \left( W(g_\rho(y)) + |g_\rho'(y)|^2 \right) \, dy
\]
for all \( n \) sufficiently large.

Since \( W(\beta) = 0 \), by the mean value theorem
\[
W(t) = W(\beta) + W'(\theta)(t - \beta) = 0 + W'(\theta)(t - \beta)
\]
for some \( \theta \) between \( t \) and \( \beta \). Hence, in \([b_\rho, b_\rho + 1]\),
\[
W(g_\rho(y)) \leq (\beta - g_\rho(y)) \max_{[\alpha, \beta]} |W'| \leq (- (\rho(y - b_\rho) - \rho)) \max_{[\alpha, \beta]} |W'| \leq 2\rho \max_{[\alpha, \beta]} |W'|,
\]
\[
g_\rho'(y) = \rho.
\]
It follows that
\[
\int_{a_\rho - 1}^{b_\rho + 1} \left( W(g_\rho(y)) + |g_\rho'(y)|^2 \right) \, dy \leq C\rho,
\]
where we have made the change of variables $s = w_n(y)$. On the other hand,

$$
\int_{a_n}^{b_n} \left( W(g_n(y)) + |g'_n(y)|^2 \right) \, dx = \int_{a_n}^{b_n} \left( W(g(y)) + |g'(y)|^2 \right) \, dx \\
\leq \int_{\mathbb{R}} (W(g(y)) + |g'(y)|^2) \, dy = c_W.
$$

Hence, we get

$$
\int_{\Omega} \left( \frac{1}{\varepsilon_n} W(u_{n,\rho}) + \varepsilon_n |u'_{n,\rho}|^2 \right) \, dx \leq c_W \rho + C \rho m.
$$

In turn,

$$\limsup_{\rho \to 0} \limsup_{n \to \infty} \int_{\Omega} \left( \frac{1}{\varepsilon_n} W(u_{n,\rho}) + \varepsilon_n |u'_{n,\rho}|^2 \right) \, dx \leq c_W \text{ (number of jumps of } u).$$

Moreover, using the change of variables $\frac{t_i + r}{\varepsilon_n} = y$ or $\frac{t_i - r}{\varepsilon_n} = y$,

$$
\int_{\Omega} |u_{n,\rho} - u| \, dx = \sum_{i=1}^{m} \int_{t_i - r}^{t_i + r} |u_{n,\rho}(x) - u(x)| \, dx = \sum_{i=1}^{m} \varepsilon_n \int_{-r/\varepsilon_n}^{r/\varepsilon_n} |g_{\rho}(y) - v(y)| \, dy \\
= \sum_{i=1}^{m} \varepsilon_n \int_{a_n}^{b_n} |g_{\rho}(y) - v(y)| \, dy \leq C_\rho \varepsilon_n \to 0
$$

as $n \to \infty$, where

$$v(y) := \begin{cases} 
\beta & \text{if } y > 0, \\
\alpha & \text{if } y \leq 0.
\end{cases}$$

We now diagonalize the sequence $\{u_{n,\rho}\}$ to obtaining a sequence $\{u_{n,\rho_n}\}$ converging to $u$ in $L^1(\Omega)$ and such that

$$\lim_{n \to \infty} \int_{\Omega} \left( \frac{1}{\varepsilon_n} W(u_{n,\rho_n}) + \varepsilon_n |u'_{n,\rho_n}|^2 \right) \, dx \leq c_W \text{ (number of jumps of } u).$$

7 **Liminf Inequality, $N = 1, d \geq 1$**

In this case the constant $c_W$ should be replaced by

$$c_W := \inf \left\{ \int_{\mathbb{R}} (W(g(t)) + |g'(t)|^2) \, dt : g \in H^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^d) \text{ such that } \lim_{t \to -\infty} g(t) = \alpha, \lim_{t \to \infty} g(t) = \beta \right\}.$$

We proceed as in the case $d = 1$ up to (9). Fix $\rho > 0$ small. Since $u_n(-\delta) \to \alpha$ and $u_n(\delta) \to \beta$, we have that

$$|v_n(-\delta/\varepsilon_n) - \alpha| = |u_n(-\delta) - \alpha| < \rho, \quad |v_n(\delta/\varepsilon_n) - \beta| = |u_n(\delta) - \beta| < \rho$$
for all $n$ large.

We now extend $v_n$ to $\mathbb{R}$ by setting

$$w_n(y) := \begin{cases} 
\beta & \text{if } y \geq \delta/\varepsilon_n + 1, \\
(\beta - v_n(\delta/\varepsilon_n))(y - \delta/\varepsilon_n) + v_n(\delta/\varepsilon_n) & \text{if } \delta/\varepsilon_n < y < \delta/\varepsilon_n + 1, \\
v_n(y) & \text{if } -\delta/\varepsilon_n \leq y \leq \delta/\varepsilon_n, \\
(v_n(-\delta/\varepsilon_n) - \alpha)(y + \delta/\varepsilon_n) + v_n(-\delta/\varepsilon_n) & \text{if } -\delta/\varepsilon_n - 1 < y < -\delta/\varepsilon_n, \\
\alpha & \text{if } y \leq -\delta/\varepsilon_n - 1.
\end{cases}$$

Since $W(\beta) = 0$ by the mean value theorem

$$W(z) = W(\beta) + \nabla W(\theta) \cdot (z - \beta) = 0 + \nabla W(\theta) \cdot (z - \beta)$$

for some $\theta$ in the segment between $z$ and $\beta$. Hence, in $[\delta/\varepsilon_n, \delta/\varepsilon_n + 1]$,

$$W(w_n(y)) \leq |v_n(y) - \beta| \max_{B(\beta, \varepsilon)} |\nabla W| \leq C \rho,$$

$$w'_n(y) = \rho.$$

A similar estimate can be made in the interval $[-\delta/\varepsilon_n - 1, -\delta/\varepsilon_n]$. It follows that

$$\int_{-\delta/\varepsilon_n}^{\delta/\varepsilon_n} (W(v_n(y)) + |v'_n(y)|^2) \, dy \geq \int_{\mathbb{R}} (W(w_n) + |w'_n|^2) \, dy - C \rho,$$

$$\geq c_W - C \rho.$$  

In turn,

$$\liminf_{n \to \infty} \int_{\Omega} \left( \frac{1}{\varepsilon_n} W(u_n) + \varepsilon_n |u'_n|^2 \right) \, dx \geq c_W \text{ (number of jumps of } u) - C \rho.$$

Letting $\rho \to 0$ gives

$$\liminf_{n \to \infty} \int_{\Omega} \left( \frac{1}{\varepsilon_n} W(u_n) + \varepsilon_n |u'_n|^2 \right) \, dx \geq c_W \text{ (number of jumps of } u).$$

8 **Limsup Inequality, $N = 1, d \geq 1$**

Fix $\eta > 0$ and by (13) find $g \in H^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^d)$ such that $\lim_{t \to \infty} g(t) = \alpha$, $\lim_{t \to -\infty} g(t) = \beta$ and

$$\int_{\mathbb{R}} \left( W(g(t)) + |g'(t)|^2 \right) \, dt \leq c_W + \eta. \quad (14)$$
Let \( u \) be as in the previous section. Fix \( \rho > 0 \) and let \( b_\rho \gg 1 \) be such that 
\(|g(b_\rho) - \beta| < \rho\) and let \( a_\rho \ll -1 \) be such that 
\(|g(a_\rho) - \alpha| < \rho\). Define

\[
g_\rho(y) := \begin{cases} 
\beta & \text{if } y \geq b_\rho + 1, \\
(\beta - g(b_\rho))(y - b_\rho) + g(b_\rho) & \text{if } b_\rho < y < b_\rho + 1, \\
g(y) & \text{if } a_\rho \leq y \leq b_\rho, \\
(g(a_\rho) - \alpha)(y - a_\rho) + g(a_\rho) & \text{if } a_\rho - 1 < y < a_\rho, \\
\alpha & \text{if } y \leq a_\rho - 1,
\end{cases}
\] (15)

and

\[
u_{n, \rho}(x) := \begin{cases} 
g_\rho\left(\frac{x - t_{2i}}{\varepsilon_n}\right) & \text{if } t_{2i} - r < x < t_{2i} + r, \\
g_\rho\left(\frac{t_{2i+1} - x}{\varepsilon_n}\right) & \text{if } t_{2i+1} - r < x < t_{2i+1} + r, \\
u(x) & \text{otherwise},
\end{cases}
\]

where \( 2r < \min\{t_{i+1} - t_i : i = 0, \ldots, m - 1\} \). Reasoning as in the previous sections we get

\[
\int_{\Omega} \left( \frac{1}{\varepsilon_n} W(u_{n, \rho}) + \varepsilon_n |u_{n, \rho}'|^2 \right) \, dx = \sum_{i=1}^{m} \int_{t_{i-1}-r}^{t_i+r} \left( \frac{1}{\varepsilon_n} W(u_{n, \rho}) + \varepsilon_n |u_{n, \rho}'|^2 \right) \, dx \\
= \sum_{i=1}^{m} \int_{r/a_n}^{r/a_n} \left( W(g_\rho(y)) + |g_\rho'(y)|^2 \right) \, dx \\
= \sum_{i=1}^{m} \int_{b_\rho+1}^{b_\rho-1} \left( W(g_\rho(y)) + |g_\rho'(y)|^2 \right) \, dx \\
\leq m \int_{\mathbb{R}} \left( W(g(t)) + |g'(t)|^2 \right) \, dt + C_\rho \\
\leq (c_W + \eta) m + C_\rho
\]

for all \( n \) sufficiently large. Note that we can take \( \eta = \rho \). It follows that

\[
\limsup_{\rho \to 0^+} \limsup_{n \to \infty} \int_{\Omega} \left( \frac{1}{\varepsilon_n} W(u_{n, \rho}) + \varepsilon_n |u_{n, \rho}'|^2 \right) \, dx \leq c_W m
\]

and as in (12), \( \int_{\Omega} |u_{n, \rho} - u| \, dx \to 0 \) as \( n \to \infty \). Again we can diagonalize to get

a sequence \( \{u_{n, \rho_n}\} \).

9 \ Liminf Inequality, \( N \geq 1, \, d = 1 \)

10 \ Liminf Inequality, \( N \geq 1, \, d \geq 1 \)

11 \ Limsup Inequality, \( N \geq 1, \, d \geq 1 \)

For \( N \geq 1 \), given \( u \in BV(\Omega; \{\alpha, \beta\}) \), we write

\[
u = \alpha \chi_E + \beta \chi_{\Omega \setminus E}.
\]
Assume first that $E$ is a regular set, that is, that $E$ is an open set with $\Omega \cap \partial E$ of class $C^2$, and that $E$ meets the boundary of $\Omega$ transversally, that is, $\mathcal{H}^{N-1}(\partial E \cap \partial \Omega) = 0$. Let $g$ and $g_\rho$ be as in (14) and (15). In this case we take
\begin{equation}
 u_{n, \rho} (x) := \begin{cases} 
 \alpha & \text{if } \text{dist} (x, \partial E) < -\varepsilon_n L_\rho \\
 g_\rho \left( \text{dist} (x, \partial E) / \varepsilon_n \right) & \text{if } |\text{dist} (x, \partial E)| \leq \varepsilon_n L_\rho, \\
 \beta & \text{if } \text{dist} (x, \partial E) > \varepsilon_n L_\rho,
\end{cases}
\end{equation}
where $\text{dist}$ is the signed distance, and $L_\rho > 0$ is such that $g_\rho (L_\rho) = \beta$ and $g_\rho (-L_\rho) = \alpha$. Note that the function $\text{dist} (x, \partial E)$ is of class $C^2$ in the set
\[ \{x \in \mathbb{R}^N : |\text{dist} (x, \partial E)| < r \} \]
for $r$ small, with $|\nabla \text{dist} (x, \partial E)| = 1$. Moreover,
\[ \lim_{r \to 0^+} \mathcal{H}^{N-1} \left( \{x \in \mathbb{R}^N : \text{dist} (x, \partial E) = r\} \right) = \mathcal{H}^{N-1} (\partial E). \]
Hence, by the coarea formula
\begin{equation}
\int_\Omega \left( \frac{1}{\varepsilon_n} W(u_{n, \rho}) + \varepsilon_n |\nabla u_{n, \rho}|^2 \right) \, dx \\
= \frac{1}{\varepsilon_n} \int_{|\text{dist}(x, \partial E)| \leq \varepsilon_n L_\rho} \left( W(g_\rho (\text{dist} (x, \partial E) / \varepsilon_n)) + |g_\rho' (\text{dist} (x, \partial E) / \varepsilon_n)|^2 \right) \, dx \\
= \frac{1}{\varepsilon_n} \int_{-\varepsilon_n L_\rho}^{\varepsilon_n L_\rho} \left( W(g_\rho (r / \varepsilon_n)) + |g_\rho' (r / \varepsilon_n)|^2 \right) \, H^{N-1} \left( \{x \in \mathbb{R}^N : \text{dist} (x, \partial E) = r\} \right) \, dr \\
= \int_{-L_\rho}^{L_\rho} \left( W(g_\rho (r)) + |g_\rho' (r)|^2 \right) \, H^{N-1} \left( \{x \in \mathbb{R}^N : \text{dist} (x, \partial E) = \varepsilon_n r\} \right) \, dr \\
\rightarrow \int_{-L_\rho}^{L_\rho} \left( W(g_\rho (r)) + |g_\rho' (r)|^2 \right) \, dH^{N-1} (\partial E). 
\end{equation}
In turn,
\begin{equation}
\limsup_{n \to \infty} \int_\Omega \left( \frac{1}{\varepsilon_n} W(u_{n, \rho}) + \varepsilon_n |\nabla u_{n, \rho}|^2 \right) \, dx \leq \int_{\mathbb{R}} \left[ W \left( \frac{d}{dt} \right) + \left| \frac{d'}{dt} \right| \right] \, dt \, H^{N-1} (\partial E) \\
+ C \rho H^{N-1} (\partial E), 
\end{equation}
and so taking $\eta = \rho$ and using (14),
\begin{equation}
\limsup_{\rho \to 0^+} \limsup_{n \to \infty} \int_\Omega \left( \frac{1}{\varepsilon_n} W(u_{n, \rho}) + \varepsilon_n |\nabla u_{n, \rho}|^2 \right) \, dx \leq c_{\rho} H^{N-1} (\partial E). 
\end{equation}