Pattern Formation and Partial Differential Equations

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Abstract: In this course, I will discuss three partial differential equations (PDE) that model pattern formation. Numerical simulations reveal that solutions of these deterministic equations have indeed stationary or self-similar statistics, which are independent of the system size and of the details of the initial data. We show how PDE methods can be used to understand some aspects of this universal behavior.

The first PDE has the structure of a gradient flow (a feature on which the analysis relies), the second PDE has the structure of a *driven* gradient flow, whereas the third PDE is half-way between a conservative and a dissipative system.

1. Bounds on the coarsening rate in spinodal decomposition

The PDE — the Cahn-Hilliard equation — is given by

$$\partial_t u + \triangle(u(1-u^2) + \triangle u) \;=\; 0$$

with periodic boundary conditions in the spatial domain $(0, L)^d$. Here, u denotes the (renormalized) volume fraction of a binary mixture, which is quenched (slightly) below the critical temperature and thus wants to segregate.

Numerical simulations reveal that for generic initial data (e. g. small amplitude white noise) after an initial layer, $(0, L)^d$ divides into a convoluted domain where $u \approx 1$ and its complement where $u \approx -1$, separated by a characteristic interfacial layer of width $O(1)$. This domain configuration coarsens over time. More precisely, the average length scale of the domains behaves as $O(t^{1/3})$. This is reflected by the fact that the average energy per volume, i. e. $E = L^{-d} \int \frac{1}{2}|\nabla u|^2 + \frac{1}{4}(1-u^2)^2\,dx$, behaves as $O(t^{-1/3})$.

We shall prove that, in a time-averaged sense, $E \geq O(t^{-1/3})$. This is joint work with R. V. Kohn.

2. Bounds on the Nusselt number in Rayleigh-Bénard convection

The system of PDEs is given by an advection-diffusion equation for the temperature T, and the Stokes equations with buoyancy for the fluid velocity u, i. e.

$$\begin{aligned}\partial_t T + \nabla \cdot (T u) - \triangle T &= 0, \\ -\triangle u + \nabla p &= T\,(0,0,1), \\ \nabla \cdot u &= 0\end{aligned}$$

in the 3-d spatial domain $(0, L)^2 \times (0, H)$ with periodic boundary conditions in the two horizontal dimensions. The PDE is complemented by inhomogeneous (and thus driving) Dirichlet boundary conditions at the top and bottom boundaries

$$T \;=\; 1 \;\text{ for } z = 0, \quad T \;=\; 0 \;\text{ for } z = H, \quad u \;=\; 0 \;\text{ for } z = 0, H.$$

Experiments and numerical simulations for $H, L \gg 1$ show a chaotic velocity field u, with regions of high temperature $T \approx 1$ in form of mushrooms (plumes). This leads to a high upwards heat transport — much higher than the one mediated by diffusion allone. This upwards heat flux is given by the Nusselt number $Nu := L^{-2}H^{-1} \int T\, u \cdot (1,0,0) dx$. Experiments and asymptotic analysis suggest that $Nu = O(H)$.

With the help of the background field method, we will prove that indeed $Nu \leq O(H)$ in $H \gg 1$ (up to a logarithm). This is joint work with C. Doering and M. Reznikoff-Westdickenberg.

3. Bounds on the average dissipation in the Kuramoto-Sivashinsky equation

The PDE — the Kuramoto-Sivashinsky equation — is given by

$$\partial_t u + \partial_x (\frac{1}{2} u^2) + \partial_x^2 u + \partial_x^4 u \;=\; 0$$

with periodic boundary conditions on $(0, L)$. In one particular application, u denotes the slope $\partial_x h$ of a (one-dimensional) crystal surface. The Kuramoto-Sivashinsky equation describes the evolution of the crystal surface in the presence of slope selection, curvature regularization and strong deposition — in a regime where there is no coarsening of facets. It can also be seen as a toy model for the energy transfer from large wave lengths to small wave lengths in the Navier Stokes equations.

For $L \gg 1$, numerical simulations reveal that the solutions have an average length scale of $O(1)$, and an average amplitude of $O(1)$ and display spatio-temporal chaos.

We will argue that the average dissipation rate, i. e. $\lim_{T\uparrow\infty} T^{-1}L^{-1}\int_0^T\int_0^L(\partial_x^2 u)^2 dxdt$, is $O(1)$ in $L \gg 1$ (up to a logarithm). The argument relies on a new observation on the inhomogeneous inviscid Burger's equation

$$\partial_t u + \partial_x(\frac{1}{2}u^2) \;=\; \partial_x f.$$