Blowing up the power of a singular cardinal of uncountable cofinality with collapses

Sittinon (New) Jirattikansakul

RIMS Set Theory Workshop

November 18th, 2020
Outline

- Definitions
- Main theorem
- Extenders
- Big Pictures
- Forcings
- Forcings extensions
- Some conclusions
Definitions

Definition

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Extenders $E$ on $(\kappa, \lambda)$ and $F$ on $(\kappa', \lambda)$ are coherent if $j_F(E) \upharpoonright \lambda = E$ where $j_F$ is an embedding derived from $F$. 

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From the definition above, we have that $E$ is Mitchell below $F$ in the sense that $E \in \text{Ult}(V, F)$. 
Main theorem

Theorem (J.)

Given an increasing sequence of cardinals $\langle \kappa_\alpha : \alpha < \eta \rangle$ where $\eta < \kappa_0$ is limit. Let $\lambda = (\sup_{\alpha < \eta} \kappa_\alpha)^{++}$. 
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Theorem (J.)

Given an increasing sequence of cardinals $\langle \kappa_\alpha : \alpha < \eta \rangle$ where $\eta < \kappa_0$ is limit. Let $\lambda = (\sup_{\alpha < \eta} \kappa_\alpha)^{++}$. Assume for each $\alpha$, there is a $(\kappa_\alpha, \lambda)$-extender $E_\alpha$ such that:

1. If $j: V \to M$ is an ultrapower such that $\text{crit}(j) = \kappa_\alpha$, $j(\kappa_\alpha)$ and $\text{M}$ computes cardinals correctly up to and including $\lambda$.

2. There is a function $\sigma_\alpha: \kappa_\alpha \to \kappa_\alpha$ such that $j(\sigma_\alpha(\kappa_\alpha)) = \lambda$.

3. $E_\alpha: \kappa_\alpha < \kappa_0$ is pairwise coherent.

Then there is a $\Diamond$-c.c. forcing extension such that in the generic extension, for limit $\kappa < \kappa_0$, $2^\kappa > \kappa^{++}$ and $2^{\kappa_0} = \kappa_0^{++}$.
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1. If $j_\alpha : V \to M_\alpha = \text{Ult}(V, E_\alpha)$, we have $\text{crit}(j_\alpha) = \kappa_\alpha$, $j_\alpha(\kappa_\alpha) \geq \lambda$, $\kappa_\alpha M_\alpha \subseteq M_\alpha$ and $M_\alpha$ computes cardinals correctly up to and including $\lambda$. 

2. There is a function $s_\alpha : \kappa_\alpha \to \kappa_\alpha$ such that $j_\alpha(s_\alpha(\kappa_\alpha)) = \kappa_\alpha$.

3. $\delta E_\alpha : \kappa_\alpha \times \lambda$ is pairwise coherent. Then there is a $\delta$-c.c. forcing extension such that in the generic extension, for limit $\alpha < \lambda$, $2^{\alpha(2^{\alpha} + 1)} = 2^{\lambda(2^{\lambda} + 1)}$. 

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Given an increasing sequence of cardinals $\langle \kappa_{\alpha} : \alpha < \eta \rangle$ where $\eta < \kappa_0$ is limit. Let $\lambda = (\sup_{\alpha < \eta} \kappa_{\alpha})^{++}$. Assume for each $\alpha$, there is a $(\kappa_{\alpha}, \lambda)$-extender $E_{\alpha}$ such that:

1. If $j_{\alpha} : V \to M_{\alpha} = \text{Ult}(V, E_{\alpha})$, we have $\text{crit}(j_{\alpha}) = \kappa_{\alpha}$, $j_{\alpha}(\kappa_{\alpha}) \geq \lambda$, $\kappa_{\alpha} M_{\alpha} \subseteq M_{\alpha}$ and $M_{\alpha}$ computes cardinals correctly up to and including $\lambda$.

2. There is a function $s_{\alpha} : \kappa_{\alpha} \to \kappa_{\alpha}$ such that $j_{\alpha}(s_{\alpha})(\kappa_{\alpha}) = \lambda$.

3. $\langle E_{\alpha} : \alpha < \eta \rangle$ is pairwise coherent.

Then there is a $\lambda$-c.c. forcing extension such that in the generic extension, for limit $\beta < \eta$, $2^{\aleph_{\beta}} > \aleph_{\beta+1}$ and $2^{\aleph_{\eta}} = \aleph_{\eta+2}$. 
Recall $\lambda = \sup_{\alpha < \eta} \kappa_\alpha^{++}$. 

Definition $mc(d) = (j(d))^1 = \{(j(a)), a \in d\}$. Abbreviate $mc(d)$ by $mc$. 

Definition $A_{2E}(d) = mc_{2j(A)}$. 

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$d_\alpha$ is an $\alpha$-domain if $d_\alpha \in [\lambda]^{\kappa_\alpha}$ and $\kappa_\alpha + 1 \subseteq d_\alpha$. 
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**Definition**

$mc_\alpha(d_\alpha) = (j_\alpha \upharpoonright d_\alpha)^{-1} = \{(j_\alpha(\gamma), \gamma) : \gamma \in d_\alpha\}$. 
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**Definition**

$A \in E_\alpha(d_\alpha)$ *iff* $mc_\alpha \in j_\alpha(A)$.
Extenders

Definition

$\text{OB}_\alpha(d_\alpha)$ is the collection of functions $\mu$ such that

1. $\text{dom}(\mu) \subseteq d_\alpha$, $\text{rge}(\mu) \subseteq \mathcal{P}(\alpha)$, and $\alpha \notin \text{rge}(\mu)$,
2. $\text{dom}(\mu) | \text{dom}(\mu) = \mu(\mathcal{P}(\alpha))$, which is below $\mathcal{P}(\alpha)$, and $\mu(\mathcal{P}(\alpha))$ is inaccessible,
3. $\text{dom}(\mu) \setminus \mathcal{P}(\alpha) = \mu(\mathcal{P}(\alpha))$.
4. $\mu$ is order-preserving.
5. For $\beta \in \text{dom}(\mu) \setminus \mathcal{P}(\alpha)$, $\mu(\beta) = \beta$. 

Lemma $\text{OB}_\alpha(d_\alpha) \subseteq \mathcal{E}_\alpha(d_\alpha)$. 

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3. \( \text{dom}(\mu) \cap \kappa_\alpha = \mu(\kappa_\alpha). \)
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1. \( \text{dom}(\mu) \subseteq d_\alpha, \text{rge}(\mu) \subseteq \kappa_\alpha, \text{ and } \kappa_\alpha \in \text{dom}(\mu). \)
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4. \( \mu \text{ is order-preserving}. \)
5. For \( \beta \in \text{dom}(\mu) \cap \kappa_\alpha, \mu(\beta) = \beta. \)

Lemma

\( \text{OB}_\alpha(d_\alpha) \subseteq E_\alpha(d_\alpha). \)
Recall $mc_\alpha = \{(j_\alpha(\gamma), \gamma) : \gamma \in d_\alpha\}$. Also $(j_\alpha(\kappa_\alpha), \kappa_\alpha) \in mc_\alpha$ because $\kappa_\alpha \in d_\alpha$. 
Recall $mc_\alpha = \{(j_\alpha(\gamma), \gamma) : \gamma \in d_\alpha\}$. Also $(j_\alpha(\kappa_\alpha), \kappa_\alpha) \in mc_\alpha$ because $\kappa_\alpha \in d_\alpha$.

**Proof.**

1. $(\text{dom}(\mu) \subseteq d_\alpha, \text{rge}(\mu) \subseteq \kappa_\alpha, \text{and } \kappa_\alpha \in \text{dom}(\mu))$. 
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Proof.

1. $(\text{dom}(\mu) \subseteq d_\alpha$, $\text{rge}(\mu) \subseteq \kappa_\alpha$, and $\kappa_\alpha \in \text{dom}(\mu))$.

   $\text{dom}(mc_\alpha) = j_\alpha[d_\alpha] \subseteq j_\alpha(d_\alpha)$. $\text{rge}(mc_\alpha) = d_\alpha \subseteq \lambda \subseteq j_\alpha(\kappa_\alpha)$. 

   

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2. $(|\text{dom}(\mu)| = \mu(\kappa_\alpha))$. $|\text{dom}(mc_\alpha)| = \kappa_\alpha = mc_\alpha(j_\alpha(\kappa_\alpha))$. 
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The rests are straightforward.

If $d_\alpha \subseteq d'_\alpha$, we have a natural projection $\pi_{d'_\alpha, d_\alpha} : \mu \mapsto \mu \upharpoonright d_\alpha$. 
Recall $\text{mc}_\alpha = \{(j_\alpha(\gamma), \gamma) : \gamma \in d_\alpha\}$. Also $(j_\alpha(\kappa_\alpha), \kappa_\alpha) \in \text{mc}_\alpha$ because $\kappa_\alpha \in d_\alpha$.

**Proof.**

1. $(\text{dom}(\mu) \subseteq d_\alpha, \text{rge}(\mu) \subseteq \kappa_\alpha$, and $\kappa_\alpha \in \text{dom}(\mu))$.
   
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2. $(|\text{dom}(\mu)| = \mu(\kappa_\alpha)) |\text{dom}(\text{mc}_\alpha)| = \kappa_\alpha = \text{mc}_\alpha(j_\alpha(\kappa_\alpha))$.

The rests are straightforward.

If $d_\alpha \subseteq d'_\alpha$, we have a natural projection $\pi_{d'_\alpha,d_\alpha} : \mu \mapsto \mu \upharpoonright d_\alpha$. This induces a projection from $E_\alpha(d'_\alpha)$ to $E_\alpha(d_\alpha)$. 
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A condition is of the form $p = \langle p_\alpha : \alpha < \eta \rangle$ such that for each $\alpha$, if $p_\alpha$ is pure, then

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- $p_\alpha$ will have 3 parts: $f_\alpha$-part, $A_\alpha$-part, and $\tilde{H}_\alpha$-part.
- $f_\alpha$ lives in a Cohen forcing whose domain is an $\alpha$-domain $d_\alpha$, range is a subset of $\kappa_\alpha$. 

If $p_\alpha$ is impure, then $p_\alpha$ will have 3 parts: $f_\alpha$-part, $A_\alpha$-part, and $\tilde{H}_\alpha$-part. $f_\alpha$ lives in a Cohen forcing.
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Big pictures

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Instead of giving a formal definition, we start off with a pure condition. A pure condition is $p = \langle p_\alpha : \alpha < \eta \rangle$ such that $p_\alpha = \langle f_\alpha, A_\alpha, \check{H}_\alpha \rangle$ such that

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2. $A_\alpha \in E_\alpha(d_\alpha)$.
3. $\check{H}_\alpha = \langle H^0_\alpha, H^1_\alpha, H^2_\alpha \rangle$ where $\text{dom}(H^l_\alpha)$ depends on the measure-one set $A_\alpha$.
4. $\langle d_\alpha : \alpha < \eta \rangle$ is $\subseteq$-increasing.
5. ...
Forcing extensions

\[ p_0 = \langle f_0, A_0, \tilde{H}_0 \rangle \quad p_1 = \langle f_1, A_1, \tilde{H}_1 \rangle \quad p_2 = \langle f_2, A_2, \tilde{H}_2 \rangle \quad p_3 = \langle f_3, A_3, \tilde{H}_3 \rangle \]

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Direct extension: \( q \preceq^* p \) if for all \( \alpha \) we have
Forcing extensions

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Direct extension: \( q \leq^* p \) if for all \( \alpha \) we have

\[ g_\alpha \leq f_\alpha. \]
Forcing extensions

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Direct extension: \( q \leq^* p \) if for all \( \alpha \) we have

1. \( g_\alpha \leq f_\alpha \).
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   \( \{ \mu \upharpoonright \text{dom}(f_\alpha) : \mu \in B_\alpha \} \subseteq A_\alpha \).
Forcing extensions

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3. For \( l = 0, 1, 2 \), \( K_\alpha^l(\mu) \leq H_\alpha^l(\mu \upharpoonright \text{dom}(f_\alpha)) \).
Forcing extensions

\[ \begin{align*}
0 & \quad | & \quad 1 & \quad | & \quad 2 & \quad | & \quad 3 \\
\quad & \quad | & \quad & \quad | & \quad & \quad | \\
0 & \quad & \quad & \quad & \quad & \quad & \quad \\
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One-step extension (example): \( p \) is pure and \( \mu \in A_2 \).
Forcing extensions

One-step extension (example): $p$ is pure and $\mu \in A_2$. One-step extension of $p$ by $\mu$ is a condition $q$ such that:

1. $q_\alpha = p_\alpha$ for $\alpha > 2$
Forcing extensions

0

1

2

3

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Forcing extensions

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Forcing extensions

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5. \( t_0 = f_0 \circ \mu^{-1}, t_1 = f_1 \circ \mu^{-1}, C_0 = A_0 \circ \mu^{-1}, C_1 = A_1 \circ \mu^{-1} \).
Forcing extensions

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- \( \kappa_1 < \lambda_2 < \kappa_2 \).
- \( \langle q_0, q_1 \rangle \) will now live in \( \mathbb{P}^\mathcal{P}(E_0 | \lambda_2, E_1 | \lambda_2) \).
Forcing extensions

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- \( \langle q_0, q_1 \rangle \) will now live in \( \mathbb{P}^{\langle E_0 \upharpoonright \lambda_2, E_1 \upharpoonright \lambda_2 \rangle} \).
- \( \tilde{h}_2 \in \text{Col}(\kappa_1, < g_2(\kappa_2)) \times \text{Col}(g_2(\kappa_2), s_2(g_2(\kappa_2))^+) \times \text{Col}((s_2(g_2(\kappa_2)))^{+3}, < \kappa_2) \).
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\end{align*}
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- \( \kappa_1 < \lambda_2 < \kappa_2 \).
- \( \langle q_0, q_1 \rangle \) will now live in \( \mathbb{P} \langle E_0 | \lambda_2, E_1 | \lambda_2 \rangle \).
- \( \vec{h}_2 \in \text{Col}(\kappa_1, < g_2(\kappa_2)) \times \text{Col}(g_2(\kappa_2), s_2(g_2(\kappa_2))^+) \times \text{Col}((s_2(g_2(\kappa_2)))^+^3, < \kappa_2) \).
- In particular, a few cardinals in the interval \( (\kappa_1, \kappa_2] \) are preserved.
Some conclusions

Let $\kappa_\eta = \sup_{\alpha < \eta} \kappa_\alpha$. Then $\lambda = \kappa_\eta^{++}$.

- The forcing has the Prikry property.
- Only few cardinals in $(\kappa_\alpha, \kappa_{\alpha+1}]$ are preserved, and hence $\kappa_\eta$ is a cardinal, and is equal to $\kappa_\eta$.
- Need a special argument to preserve $\kappa_\eta^+$.
- The forcing is $\lambda$-c.c., so preserves $\lambda$ and $\lambda = \aleph_{\eta+2}$ in the extension.
- One can derive a scale on $\kappa_\eta$ of length $\lambda$. Hence in the extension, $\aleph_{\eta+2} = \lambda = 2^{\kappa_\eta} = 2^{\aleph_\eta}$. 
Thank you!