Problems from Previous Tests

1. (25 pts)
   (a) Give the definition of the integral of a step function.
   (b) Let s be a step function defined on $[a, b]$, and c any real number. Prove
$$\int_a^b c \cdot s(x)\, dx = c \int_a^b s(x)\, dx$$

2. (25 points)
   Using mathematical induction prove that for integers $n \geq 4$ the inequality $2^n < n!$ holds.

3. (25 points)
   (a) Give the definition of supremum and infimum.
   (b) Prove the following: Given two nonempty subsets S and T of $\mathbf{R}$ such that
$$s \leq t$$
   for every s in S and every t in T. Then S has a supremum, and T has an infimum, and they satisfy
$$sup\, S \leq inf\, T.$$

4. (25 points)
   (a) Prove $0 \cdot a = 0$ for every real number a.
   (b) If $a \cdot b = 0$, then $a = 0$ or $b = 0$. *[Some axioms and the cancellation law for addition are listed for your convenience:*
   A1: $x + y = y + x \qquad xy = yx$
   A2: $x + (y + z) = (x + y) + z \qquad x(yz) = (xy)z$
   A3: $x(y + z) = xy + xz$
   A4: There exists two distinct numbers, which we denote by 0 and 1, such that, $x + 0 = x, \qquad 1 \cdot x = x$.
   A5: For every real number x*, there is a real number* y *such that* $x + y = 0$
   A6: For every real number $x \neq 0$*, there is a real number* y *such that* $xy = 1$.
   Cancellation Law: If $a + c = b + c$ *then* $a = b$. *]*

5. (25 pts)
   (a) State the definition of integrability for a function f on the interval $[a, b]$.
   (b) **Prove the following:** Let f and g be integrable on $[a, b]$ and $f(x) \leq g(x)$ for every x in $[a, b]$, then
$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

6. (25 points)
   Compute the integral
$$\int_{-\pi}^{x} |\frac{1}{2} + \cos t| dt \qquad 0 \leq x \leq \pi/2$$

7. (25 points)
   Compute

$$\lim_{x \to 0} \frac{\sin(\tan x)}{\sqrt{|x|}}$$

8. (25 points) **Definition:** (Infinite Limits) The symbolism

$$\lim_{x \to a} f(x) = +\infty$$

   means that for every $M > 0$, there is a $\delta > 0$ such that

$$f(x) > M \qquad \text{whenever} \qquad 0 < |x - a| < \delta.$$

   Prove using the definition: If $g(x) > f(x)$ for all x and $\lim_{x \to a} f(x) = +\infty$, then $\lim_{x \to a} g(x) = +\infty$

9. (20 pts) Let f be differentiable on $I \supseteq [a, b]$, and let $|f'(x)| \leq 1$ for all $x \in I$. Prove that f satisfies

$$|f(x) - f(y)| \leq |x - y| \quad \text{for all} \quad x, y \in [a, b]$$

10. (20 points) Sketch the graph of

$$f(x) = \frac{x - 1}{x^2 + 1}$$

   .

11. (20 points) Compute the partial derivative f_x, f_y for the function

$$f(x, y) = cos(sin(xy) + y)$$

12. (20 points) Use the first fundamental theorem of calculus and the chain rule to prove the following: If f is continuous, and g is differentiable, then

$$\frac{d}{dx} \int_0^{g(x)} f(t)dt = f(g(x)) \cdot g'(x)$$

   .

13. (20 points) Prove that among all rectangles inscribed in a circle, the square has the maximum area.