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Linear Square Optimal Control Problem for Stochastic Difference Equations with Unknown Parameters

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Abstract—The problems of stability and optimal control for stochastic difference equations are receiving important attention now (see, for example, [1–3]). In this paper, the optimal control in final form is obtained for optimal control problem of stochastic linear difference equation with unknown parameters and square cost functional. For stochastic functional differential equations, analogous result are obtained in [4].

Keywords—Optimal control problem, Stochastic difference equations, Unknown parameters, Square cost functional.

1. THE STATEMENT OF THE PROBLEM

Consider the optimal control problem for the stochastic difference equation

$$x_{i+1} = \sum_{j=0}^{i} A_{i-j} x_j + D_i \eta + B_i u_i + \sigma_i \xi_{i+1}, \tag{1}$$

and the cost functional

$$J(u) = \mathbf{E}\left[x_N' F x_N + \sum_{j=0}^{N-1} \left(u_j' G_j u_j + x_j' H_j x_j\right)\right]. \tag{2}$$

Here, $\mathbf{E}$ is the mathematical expectation, $\xi_1, \dots, \xi_N$ are Gaussian mutually independent random vectors $\xi_i \in R^m$, $\mathbf{E}\xi_i = 0$, $\mathbf{E}\xi_i\xi_i' = I$, I is identity matrix, η is unknown Gaussian vector, $x_i \in R^n$, $\eta \in R^r$, $u_i \in R^l$ matrices F, G_j, H_j are positive semidefinite, A_{i-j}, D_i, B_i, σ_i are arbitrary matrices with corresponding dimensions.

In this paper, our concern with respect to the optimal control problem (1),(2) is to find a control v for which the cost functional $J(u)$ is minimal: $J(v) = \inf_{u \in U} J(u)$, U is the set of admissible controls.

Let $f_i^x = \sigma\{x_0, x_1, \dots, x_i\}$ be a σ-algebra, induced by the values of x_j, $j = 0, 1, \dots, i$. It is known [5] that the optimal, in the mean square sense, estimate of the unknown parameter η is given by the conditional expectation $m_i = \mathbf{E}\{\eta/f_i^x\}$. This estimate m_i is determined by the system of equations

$$m_{i+1} = m_i + \gamma_i D_i' \left[\sigma_i\sigma_i' + D_i\gamma_i D_i'\right]^+ \left[x_{i+1} - \sum_{j=0}^{i} A_{i-j}x_j - D_i m_i - B_i u_i\right], \tag{3}$$

$$\gamma_{i+1} = \gamma_i - \gamma_i D_i' \left[\sigma_i\sigma_i' + D_i\gamma_i D_i'\right]^+ D_i\gamma_i, \tag{4}$$

where $\gamma_i = \mathbf{E}\{(\eta - m_i)(\eta - m_i)'/f_i^x\}$, A^+ is the pseudoinverse of the matrix A. We remark that if the matrix A has the inverse matrix A^{-1}, then $A^+ = A^{-1}$.

In this way, the optimal control problem (1),(2) reduces to the optimal control problem (1)–(4).

THEOREM 1. *Let there exist a nonnegative functional* $V_i = V_i(x_0, \dots, x_i, y_i)$, *and a control* $v_i = v_i(x_0, \dots, x_i, y_i)$ *such that*

$$\inf_{u\in U} \mathbf{E}\left[\Delta V_i\left(x_0^u, \dots, x_i^u, m_i^u\right) + u_i' G_i u_i + \left(x_i^u\right)' H_i x_i^u\right]$$
$$= \mathbf{E}\left[\Delta V_i\left(x_0^v, \dots, x_i^v, m_i^v\right) + v_i' G_i v_i + (x_i^v)' H_i x_i^v\right] = 0, \tag{5}$$

$$V_N\left(x_0, \dots, x_N, y_N\right) = x_N' F x_N, \tag{6}$$

where x_i^u *and* m_i^u *is the solution of the system (1),(3),(4) with the control* $u = u_i$,

$$\Delta V_i\left(x_0^u, \dots, x_i^u, m_i^u\right) = V_{i+1}\left(x_0^u, \dots, x_{i+1}^u, m_{i+1}^u\right) - V_i\left(x_0^u, \dots, x_i^u, m_i^u\right).$$

Then v_i *is the optimal control of the problem (1),(2), and* $J(v) = \mathbf{E}V_0(x_0, m_0)$.

PROOF. From (5), it follows that

$$\sum_{i=0}^{N-1} \mathbf{E}\left[\Delta V_i\left(x_0^u, \dots, x_i^u, m_i^u\right) + u_i' G_i u_i + (x_i^u)' H_i x_i^u\right] \ge 0,$$
$$\sum_{i=0}^{N-1} \mathbf{E}\left[\Delta V_i\left(x_0^v, \dots, x_i^v, m_i^v\right) + v_i' G_i v_i + (x_i^v)' H_i x_i^v\right] = 0.$$

From this, and by virtue of (2),(6), we obtain $J(u) \ge \mathbf{E}V_0(x_0, m_0) = J(v)$. This completes the proof.

In the following sections, we will construct the functional $V_i = V_i(x_0, \dots, x_i, y_i)$, and the optimal control $v_i = v_i(x_0, \dots, x_i, y_i)$ in final form.

2. THE OPTIMAL CONTROL CONSTRUCTION

We will construct the functional $V_i = V_i(x_0, \dots, x_i, y_i)$ satisfying the conditions (5),(6) in the form

$$\begin{aligned} V_i &= x_i' P_0(i) x_i + x_i' P_1(i) y_i + y_i' P_1'(i) x_i + y_i' P_2(i) y_i + P_3(i) \\ &+ \sum_{j=0}^{i-1}\sum_{k=0}^{i-1} x_j' R(i,j,k) x_k + \sum_{j=0}^{i-1} x_i' Q_0(i,j) x_j + \sum_{j=0}^{i-1} x_j' Q_0'(i,j) x_i \\ &+ \sum_{j=0}^{i-1} y_i' Q_1(i,j) x_j + \sum_{j=0}^{i-1} x_j' Q_1'(i,j) y_i. \end{aligned} \tag{7}$$

Here it is assumed that $P_0(i)$, $P_1(i)$, and $P_2(i)$ are matrices of dimensions $n \times n$, $n \times r$, and $r \times r$, respectively, $P_3(i) \ge 0$, the matrix

$$\begin{pmatrix} P_0(i) & P_1(i) \\ P_1'(i) & P_2(i) \end{pmatrix}$$

is positive semidefinite, and $P_0'(i) = P_0(i)$, $P_2'(i) = P_2(i)$, $R'(i,j,k) = R(i,k,j)$. We shall follow the convention that $\sum_{j=0}^{-1} = 0$,

$$\begin{gathered} P_0(N) = F, \qquad P_1(N) = 0, \qquad P_2(N) = 0, \qquad P_3(N) = 0, \\ R(N,j,k) = 0, \qquad Q_0(N,j) = 0, \qquad Q_1(N,j) = 0. \end{gathered} \tag{8}$$

On substituting (7) in (5), we can calculate $\mathbf{E}\Delta V_i$. For this, setting $x_i^u = x_i$ for the first summand of V_i, we obtain

$$\begin{aligned}
&\mathbf{E}\left[x_{i+1}'P_0(i+1)x_{i+1} - x_i'P_0(i)x_i\right] \\
&= \mathbf{E}\left[\left(\sum_{j=0}^{i} A_{i-j}x_j + D_i(\eta - m_i) + D_i m_i + B_i u_i + \sigma_i\xi_{i+1}\right)' P_0(i+1)\right. \\
&\quad \times \left.\left(\sum_{k=0}^{i} A_{i-k}x_k + D_i(\eta - m_i) + D_i m_i + B_i u_i + \sigma_i\xi_{i+1}\right) - x_i'P_0(i)x_i\right] \\
&= \mathbf{E}\left[\sum_{j=0}^{i}\sum_{k=0}^{i} x_j'A_{i-j}'P_0(i+1)A_{i-k}x_k + 2\sum_{k=0}^{i} m_i'D_i'P_0(i+1)A_{i-k}x_k\right. \\
&\quad + 2\sum_{k=0}^{i} u_i'B_i'P_0(i+1)A_{i-k}x_k + m_i'D_i'P_0(i+1)D_i m_i + u_i'B_i'P_0(i+1)B_i u_i \\
&\quad \left. + \operatorname{Tr}\left[D_i'P_0(i+1)D_i\gamma_i + \sigma_i'P_0(i+1)\sigma_i\right] + 2u_i'B_i'P_0(i+1)D_i m_i - x_i'P_0(i)x_i\right] \\
&= \mathbf{E}\left[\sum_{j=0}^{i-1}\sum_{k=0}^{i-1} x_j'A_{i-j}'P_0(i+1)A_{i-k}x_k + 2\sum_{j=0}^{i-1} x_j'A_{i-j}'P_0(i+1)A_0 x_i\right. \\
&\quad + x_i'\left(A_0'P_0(i+1)A_0 - P_0(i)\right)x_i + m_i'D_i'P_0(i+1)D_i m_i \\
&\quad + 2m_i'D_i'P_0(i+1)A_0 x_i + 2\sum_{k=0}^{i-1} m_i'D_i'P_0(i+1)A_{i-k}x_k + u_i'B_i'P_0(i+1)B_i u_i \\
&\quad \left. + 2u_i'B_i'P_0(i+1)\left(D_i m_i + \sum_{k=0}^{i} A_{i-k}x_k\right) + \operatorname{Tr}\left[D_i'P_0(i+1)D_i\gamma_i + \sigma_i'P_0(i+1)\sigma_i\right]\right].
\end{aligned}$$

From (1),(3) it follows that

$$m_{i+1} = m_i + W_i\left[D_i(\eta - m_i) + \sigma_i\xi_{i+1}\right], \qquad W_i = \gamma_i D_i'\left[\sigma_i\sigma_i' + D_i\gamma_i D_i'\right]^+.$$

Therefore, for the second and third summands, we obtain

$$\begin{aligned}
&\mathbf{E}\left[x_{i+1}'P_1(i+1)m_{i+1} - x_i'P_1(i)m_i\right] \\
&= \mathbf{E}\left[\left(\sum_{j=0}^{i} A_{i-j}x_j + D_i(\eta - m_i) + D_i m_i + B_i u_i + \sigma_i\xi_{i+1}\right)' P_1(i+1)\right. \\
&\quad \times \left.\left(m_i + W_i\left[D_i(\eta - m_i) + \sigma_i\xi_{i+1}\right]\right) - x_i'P_1(i)m_i\right] \\
&= \mathbf{E}\left[\sum_{j=0}^{i-1} x_j'A_{i-j}'P_1(i+1)m_i + m_i'D_i'P_1(i+1)m_i + u_i'B_i'P_1(i+1)m_i\right. \\
&\quad \left. + \operatorname{Tr}\left[D_i'P_1(i+1)W_iD_i\gamma_i + \sigma_i'P_1(i+1)W_i\sigma_i\right] + x_i'\left(A_0'P_1(i+1) - P_1(i)\right)m_i\right].
\end{aligned}$$

Analogously, for the fourth summand, we get

$$\begin{aligned}
&\mathbf{E}\left[m'_{i+1}P_2(i+1)m_{i+1} - m'_iP_2(i)m_i\right] \\
&\quad = \mathbf{E}\left[(m_i + W_i[D_i(\eta - m_i) + \sigma_i\xi_{i+1}])' P_2(i+1)(m_i \right. \\
&\qquad \left. + W_i\left[D_i(\eta - m_i) + \sigma_i\xi_{i+1}\right]) - m'_iP_2(i)m_i\right] \\
&\quad = \mathbf{E}\left[m'_i\Delta P_2(i)m_i + \operatorname{Tr}\left[D'_iW'_iP_2(i+1)W_iD_i\gamma_i + \sigma'_iW'_iP_2(i+1)W_i\sigma_i\right]\right].
\end{aligned}$$

Let $\Delta R(i,j,k) = R(i+1,j,k) - R(i,j,k)$. Then,

$$\begin{aligned}
&\mathbf{E}\left[\sum_{j=0}^{i}\sum_{k=0}^{i} x'_jR(i+1,j,k)x_k - \sum_{j=0}^{i-1}\sum_{k=0}^{i-1} x'_jR(i,j,k)x_k\right] \\
&\quad = \mathbf{E}\left[\sum_{j=0}^{i-1}\left(\sum_{k=0}^{i-1} x'_jR(i+1,j,k)x_k + x'_jR(i+1,j,i)x_i\right)\right. \\
&\qquad \left. + \sum_{k=0}^{i-1} x'_iR(i+1,i,k)x_k + x'_iR(i+1,i,i)x_i - \sum_{j=0}^{i-1}\sum_{k=0}^{i-1} x'_jR(i,j,k)x_k\right] \\
&\quad = \mathbf{E}\left[\sum_{j=0}^{i-1}\sum_{k=0}^{i-1} x'_j\Delta R(i,j,k)x_k + 2\sum_{j=0}^{i-1} x'_jR(i+1,j,i)x_i + x'_iR(i+1,i,i)x_i\right].
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&\mathbf{E}\left[\sum_{j=0}^{i} x'_{i+1}Q_0(i+1,j)x_j - \sum_{j=0}^{i-1} x'_iQ_0(i,j)x_j\right] \\
&\quad = \mathbf{E}\left[\sum_{j=0}^{i}\left(\sum_{k=0}^{i} A_{i-k}x_k + D_i\eta_i + B_iu_i + \sigma_i\xi_{i+1}\right)' Q_0(i+1,j)x_j - \sum_{j=0}^{i-1} x'_iQ_0(i,j)x_j\right] \\
&\quad = \mathbf{E}\left[\sum_{j=0}^{i}\sum_{k=0}^{i} x'_kA'_{i-k}Q_0(i+1,j)x_j + \sum_{j=0}^{i} m'_iD'_iQ_0(i+1,j)x_j\right. \\
&\qquad \left. + \sum_{j=0}^{i} u'_iB'_iQ_0(i+1,j)x_j - \sum_{j=0}^{i-1} x'_iQ_0(i,j)x_j\right] \\
&\quad = \mathbf{E}\left[\sum_{j=0}^{i-1}\sum_{k=0}^{i-1} x'_kA'_{i-k}Q_0(i+1,j)x_j\right. \\
&\qquad + \sum_{j=0}^{i-1} x'_i\left(A'_0Q_0(i+1,j) + Q'_0(i+1,i)A_{i-j} - Q_0(i,j)\right)x_j \\
&\qquad \left. + \sum_{j=0}^{i-1} m'_iD'_iQ_0(i+1,j)x_j + m'_iD'_iQ_0(i+1,i)x_i + u'_iB'_i\sum_{j=0}^{i} Q_0(i+1,j)x_j\right].
\end{aligned}$$

Let $\Delta Q_1(i,j) = Q_1(i+1,j) - Q_1(i,j)$. Then,

$$\mathbf{E}\left[\sum_{j=0}^{i} m'_{i+1}Q_1(i+1,j)x_j - \sum_{j=0}^{i-1} m'_iQ_1(i,j)x_j\right]$$

$$= \mathbf{E}\left[\sum_{j=0}^{i}(m_i + W_i[D_i(\eta - m_i) + \sigma_i\xi_{i+1}])' Q_1(i+1,j)x_j - \sum_{j=0}^{i-1} m_i' Q_1(i,j)x_j\right]$$
$$= \mathbf{E}\left[m_i' Q_1(i+1,i)x_i + \sum_{j=0}^{i-1} m_i' \Delta Q_1(i,j)x_j\right].$$

As a result from (5), we have

$$\begin{aligned}
\inf_{u\in U} \mathbf{E} \Bigg[&\sum_{j=0}^{i-1}\sum_{k=0}^{i-1} x_j' \left(A_{i-j}' P_0(i+1)A_{i-k} + \Delta R(i,j,k) + A_{i-j}' Q_0(i+1,k) + Q_0'(i+1,j)A_{i-k}\right) x_k \\
&+ 2\sum_{j=0}^{i-1} x_i' \left(A_0' P_0(i+1)A_{i-j} + R(i+1,i,j) + A_0' Q_0(i+1,j) + Q_0'(i+1,i)A_{i-j} - Q_0(i,j)\right) x_j \\
&+ x_i' \left(A_0' P_0(i+1)A_0 + H_i + R(i+1,i,i) - P_0(i)\right) x_i \\
&+ m_i' \left(\Delta P_2(i) + D_i' P_0(i+1)D_i + D_i' P_1(i+1) + P_1'(i+1)D_i\right) m_i \\
&+ 2x_i' \left(A_0' P_0(i+1)D_i + Q_0'(i+1,i)D_i + Q_1'(i+1,i) + A_0' P_1(i+1) - P_1(i)\right) m_i \\
&+ 2\sum_{k=0}^{i-1} m_i' \left(D_i' P_0(i+1)A_{i-k} + \Delta Q_1(i,k) + D_i' Q_0(i+1,k) + P_1'(i+1)A_{i-k}\right) x_k \\
&+ u_i' Z(i) u_i + 2u_i' B_i' \left(Z_0(i)m_i + \sum_{j=0}^{i} Z_1(i,j)x_j\right) \\
&+ \mathrm{Tr}\left[D_i' P_0(i+1)D_i\gamma_i + \sigma_i' P_0(i+1)\sigma_i\right] + 2\mathrm{Tr}\left[D_i' P_1(i+1)W_iD_i\gamma_i + \sigma_i' P_1(i+1)W_i\sigma_i\right] \\
&+ \mathrm{Tr}\left[D_i' W_i' P_2(i+1)W_iD_i\gamma_i + \sigma_i' W_i' P_2(i+1)W_i\sigma_i\right] + \Delta P_3(i)\Bigg] = 0,
\end{aligned} \tag{9}$$

where

$$\begin{aligned} Z(i) = G_i + B_i' P_0(i+1)B_i, \qquad Z_0(i) = P_0(i+1)D_i + P_1(i+1), \\ Z_1(i,j) = P_0(i+1)A_{i-j} + Q_0(i+1,j). \end{aligned} \tag{10}$$

From this, it follows that the optimal control v_i has the form

$$v_i = -Z^+(i)B_i' \left(Z_0(i)m_i + \sum_{j=0}^{i} Z_1(i,j)x_j\right). \tag{11}$$

Substituting (11),(10) into (9), we obtain the equations for $P_0(i)$, $P_1(i)$, $P_2(i)$, $P_3(i)$, $R(i,j,k)$, $Q_0(i,j)$, and $Q_1(i,j)$.

3. THE OPTIMAL COST CONSTRUCTION

Let $B_i^0 = B_iZ^+(i)B_i'$, $i = 0,1,\ldots,N-1$. Substituting the control (11) into the expression

$$u_i' Z(i)u_i + 2u_i' B_i' \left(Z_0(i)m_i + \sum_{j=0}^{i} Z_1(i,j)x_j\right),$$

we obtain

$$
\begin{aligned}
&-\left(Z_0(i)m_i + \sum_{j=0}^{i} Z_1(i,j)x_j\right)' B_i^0 \left(Z_0(i)m_i + \sum_{j=0}^{i} Z_1(i,j)x_j\right) = -m_i' Z_0'(i) B_i^0 Z_0(i) m_i \\
&- 2m_i' Z_0'(i) B_i^0 Z_1(i,i)x_i - 2\sum_{j=0}^{i-1} m_i' Z_0'(i) B_i^0 Z_1(i,j)x_j - x_i' Z_1'(i,i) B_i^0 Z_1(i,i) x_i \\
&- 2\sum_{j=0}^{i-1} x_i' Z_1'(i,i) B_i^0 Z_1(i,j) x_j - \sum_{j=0}^{i-1}\sum_{k=0}^{i-1} x_j' Z_1'(i,j) B_i^0 Z_1(i,k) x_k.
\end{aligned} \tag{12}
$$

Combining (12) with (9), we get

$$
\begin{aligned}
\mathbf{E}\Bigg[&\sum_{j=0}^{i-1}\sum_{k=0}^{i-1} x_j' \left(A_{i-j}' P_0(i+1) A_{i-k} + \Delta R(i,j,k)\right. \\
&\left.+A_{i-j}' Q_0(i+1,k) + Q_0'(i+1,j) A_{i-k} - Z_1'(i,j) B_i^0 Z_1(i,k)\right) x_k \\
&+ 2\sum_{j=0}^{i-1} x_i' \left(A_0' P_0(i+1) A_{i-j} + R(i+1,i,j) + A_0' Q_0(i+1,j) + Q_0'(i+1,i) A_{i-j}\right. \\
&\left.-Q_0(i,j) - Z_1'(i,i) B_i^0 Z_1(i,j)\right) x_j \\
&+ x_i' \left(A_0' P_0(i+1) A_0 + H_i + R(i+1,i,i) - P_0(i) - Z_1'(i,i) B_i^0 Z_1(i,i)\right) x_i \\
&+ m_i' \left(\Delta P_2(i) + D_i' P_0(i+1) D_i + D_i' P_1(i+1) + P_1'(i+1) D_i - Z_0'(i) B_i^0 Z_0(i)\right) m_i \\
&+ 2x_i' \left(A_0' P_0(i+1) D_i + Q_0'(i+1,i) D_i + Q_1'(i+1,i) + A_0' P_1(i+1) - P_1(i)\right. \\
&\left.-Z_1'(i,i) B_i^0 Z_0(i)\right) m_i + 2\sum_{k=0}^{i-1} m_i' \left(D_i' P_0(i+1) A_{i-k} + \Delta Q_1(i,k)\right. \\
&\left.+ D_i' Q_0(i+1,k) + P_1'(i+1) A_{i-k} - Z_0'(i) B_i^0 Z_1(i,k)\right) x_k \\
&+ \mathrm{Tr}\left[D_i' P_0(i+1) D_i \gamma_i + \sigma_i' P_0(i+1)\sigma_i\right] + 2\mathrm{Tr}\left[D_i' P_1(i+1) W_i D_i \gamma_i + \sigma_i' P_1(i+1) W_i \sigma_i\right] \\
&+\mathrm{Tr}\left[D_i' W_i' P_2(i+1) W_i D_i \gamma_i + \sigma_i' W_i' P_2(i+1) W_i \sigma_i\right] + \Delta P_3(i)\Bigg] = 0.
\end{aligned}
$$

Hence, we obtain the following recurrence relations for $P_0(i)$, $P_1(i)$, $P_2(i)$, $P_3(i)$, $R(i,j,k)$, $Q_0(i,j)$, and $Q_1(i,j)$:

$$P_0(i) = A_0' P_0(i+1) A_0 + R(i+1,i,i) + H_i - Z_1'(i,i) B_i^0 Z_1(i,i), \tag{13}$$

$$P_1(i) = A_0' P_1(i+1) + A_0' P_0(i+1) D_i + Q_0'(i+1,i) D_i + Q_1'(i+1,i) - Z_1'(i,i) B_i^0 Z_0(i), \tag{14}$$

$$P_2(i) = P_2(i+1) + D_i' P_0(i+1) D_i + D_i' P_1(i+1) + P_1'(i+1) D_i - Z_0'(i) B_i^0 Z_0(i), \tag{15}$$

$$
\begin{aligned}
P_3(i) = P_3(i+1) &+ \mathrm{Tr}[D_i' W_i' P_2(i+1) W_i D_i \gamma_i + \sigma_i' W_i' P_2(i+1) W_i \sigma_i + D_i' P_0(i+1) D_i \gamma_i \\
&+ \sigma_i' P_0(i+1)\sigma_i + 2D_i' P_1(i+1) W_i D_i \gamma_i + 2\sigma_i' P_1(i+1) W_i \sigma_i],
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
R(i,j,k) &= R(i+1,j,k) + A_{i-j}' P_0(i+1) A_{i-k} \\
&+ Q_0'(i+1,j) A_{i-k} + A_{i-j}' Q_0(i+1,k) - Z_1'(i,j) B_i^0 Z_1(i,k),
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
Q_0(i,j) &= A_0' P_0(i+1) A_{i-j} + A_0' Q_0(i+1,j) + Q_0'(i+1,i) A_{i-j} \\
&+ R(i+1,i,j) - Z_1'(i,i) B_i^0 Z_1(i,j),
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
Q_1(i,k) &= Q_1(i+1,k) + D_i' P_0(i+1) A_{i-k} + D_i' Q_0(i+1,k) \\
&+ P_1'(i+1) A_{i-k} - Z_0'(i) B_i^0 Z_1(i,k).
\end{aligned} \tag{19}
$$

Here,

$$i = N-1, N-2, \dots, 1, 0, \qquad j,k = 0,1,\dots,i-1.$$

By virtue of (8), (10), and (13)–(19), the functions $P_0(i)$, $P_1(i)$, $P_2(i)$, $P_3(i)$, $R(i,j,k)$, $Q_0(i,j)$, and $Q_1(i,j)$ can be calculated for all $i = 0,1,\dots,N$, $j,k = 0,1,\dots,i-1$. From this the optimal control and the optimal cost of the optimal control problem (1),(2) can be obtained by virtue of (11), (10), and (7).

4. THE SPECIAL CASE

Let $D_i = 0$. This means that unknown parameter is absent in the system (1). It is easy to see that in this case $P_1(i) = 0$, $P_2(i) = 0$, $Q_1(i,j) = 0$, and the system (13)–(19) takes the form

$$P_0(i) = A_0' P_0(i+1) A_0 + R(i+1,i,i) + H_i - Z_1'(i,i) B_i^0 Z_1(i,i), \tag{20}$$

$$P_3(i) = P_3(i+1) + \mathrm{Tr}[\sigma_i' P_0(i+1)\sigma_i], \tag{21}$$

$$\begin{aligned} R(i,j,k) = {} & R(i+1,j,k) + A_{i-j}' P_0(i+1) A_{i-k} + Q_0'(i+1,j) A_{i-k} \\ & + A_{i-j}' Q_0(i+1,k) - Z_1'(i,j) B_i^0 Z_1(i,k), \end{aligned} \tag{22}$$

$$\begin{aligned} Q_0(i,j) = {} & A_0' P_0(i+1) A_{i-j} + A_0' Q_0(i+1,j) + Q_0'(i+1,i) A_{i-j} \\ & + R(i+1,i,j) - Z_1'(i,i) B_i^0 Z_1(i,j), \end{aligned} \tag{23}$$

$$i = N-1, N-2, \dots, 1, 0, \qquad j,k = 0,1,\dots,i-1.$$

EXAMPLE. Let us consider the scalar optimal control problem

$$x_{i+1} = x_i + a x_{i-1} + b u_i + \sigma \xi_{i+1},$$

$$J(u) = \mathbf{E}\left[x_N^2 + \lambda \sum_{j=0}^{N-1} u_j^2 \right].$$

It is the special case of the problem (1),(2) with $A_0 = 1$, $A_1 = a$, $A_i = 0$, $i > 1$, $B_i = b$, $D_i = 0$, $\sigma_i = \sigma$, $F = 1$, $G_j = \lambda$, $H_j = 0$.

Solving the system (20)–(23) numerically with the values of the parameters $a = 0.5$, $b = 0.01$, $\sigma = 0.1$, $N = 10$, $\lambda = 1$, we obtain the optimal control

$$\begin{aligned} v_0 &= -0.079x_0, & v_1 &= -0.019x_0 - 0.065x_1, \\ v_2 &= -0.016x_1 - 0.054x_2, & v_3 &= -0.013x_2 - 0.045x_3, \\ v_4 &= -0.011x_3 - 0.037x_4, & v_5 &= -0.009x_4 - 0.031x_5, \\ v_6 &= -0.007x_5 - 0.025x_6, & v_7 &= -0.006x_6 - 0.020x_7, \\ v_8 &= -0.005x_7 - 0.015x_8, & v_9 &= -0.005x_8 - 0.010x_9, \end{aligned}$$

and the minimal cost

$$J(v) = \inf_{u \in U} J(u) = 5.579\mathbf{E}x_0^2 + 0.230.$$

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