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Optimal Control Problem for Nonlinear Stochastic Difference Second Kind Volterra Equations

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Abstract—The problems of stability and optimal control for stochastic difference equations are receiving important attention now (see for example [1–6]). In this paper, the necessary optimality condition for nonlinear stochastic difference second kind Volterra equation are constructed. For stochastic integral-functional equations analogous results were obtained in [7].

Keywords—Optimal control, Stochastic difference equations, Volterra difference equations.

1. THE STATEMENT OF THE PROBLEM

Consider the optimal control problem $\{x_u, J(u), \mathbf{U}\}$ where x_u is the motion trajectory, $J(u)$ is the cost functional, and $\mathbf{U}$ is the set of the admissible controls.

The control u_0, for which the cost functional $J(u)$ is minimal, i.e., $J(u_0) = \min_{u\in\mathbf{U}} J(u)$, is called optimal control.

Let u_ε, $\varepsilon \geq 0$ be the admissible control, for which there exists the limit

$$J_0(u_0) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[J(u_\varepsilon) - J(u_0)]. \tag{1}$$

Obviously, the inequality $J_0(u_0) \geq 0$ is a necessary condition for the optimality of the control u_0. The existence of the limit (1) and the form of the necessary optimality condition are depended from the form of the control u_ε. Let be, for example,

$$u_\varepsilon = u_0 + \varepsilon v, \qquad u_0, v \in \mathbf{U}. \tag{2}$$

In this case $J_0(u_0) = J_0(u_0, v)$ is Gâteaux differential.

In particular, if $J_0(u_0, v)$ is linear with respect to v then $J_0(u_0, v) = \langle J_0(u_0), v\rangle$ where $J_0(u_0)$ is Gâteaux derivative. In the linear case the inequality $\langle J_0(u_0), v\rangle \geq 0$ is equivalent to the equation $J_0(u_0) = 0$ which has the unique solution u_0 [5].

Using the calculation of the limit (1), (2) a necessary optimality condition for the nonlinear optimal control problem is obtained below.

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2. MAIN RESULT

Consider the optimal control problem for the difference equation

$$x(i+1) = \eta(i+1)+\Phi(i+1, x_{i+1})+\sum_{j=0}^{i} a(i,j,x_j,u(j))+\sum_{j=0}^{i} b(i,j,x_j,u(j))\xi(j), \quad x(0) = \eta(0), \quad (3)$$

and the cost functional

$$J(u) = \mathbf{E}\left[F(x_N) + \sum_{j=0}^{N-1} G(j, x_j, u(j))\right]. \quad (4)$$

Let $\{\Omega, \sigma, \mathbf{P}\}$ be a probability space, $i \in Z = \{0, 1, \ldots, N\}$ be a discrete time, $f_i \in \sigma$, $i \in Z$, be a family of σ-algebras, $\mathbf{E}$ be a mathematical expectation, H be a space of f_i-adapted functions $x(i) \in \mathbf{R}^n$, $i \in Z$, such that

$$\|x\|_i^2 = \max_{j \le i} \mathbf{E}|x(j)|^2 < \infty.$$

Let x_i be the trajectory of the process $x(j)$, $j \le i$, $\eta(i)$ be the f_i-adapted random values, $\|\eta\|_N < \infty$, $\xi(i)$ be f_{i+1}-adapted random values, which independent from each others and from $\eta(i)$, $\mathbf{E}\xi(i) = 0$, $\mathbf{E}|\xi(i)|^2 = 1$, $\Phi(i,\varphi) \in \mathbf{R}^n$, $a(i,j,\varphi,u) \in \mathbf{R}^n$, $b(i,j,\varphi,u) \in \mathbf{R}^n$, $0 \le j \le i \le N$, $u(i) \in \mathbf{R}^m$, $\varphi \in H$.

Arbitrary f_i-adapted function $u(i) \in \mathbf{R}^m$ with a finite norm $\|u\|_N < \infty$, is called admissible control, $\mathbf{U}$ is the set of the admissible controls.

It is supposed that $F(\varphi)$ depends from all values of the function $\varphi(j)$ by $j = 0, 1, \ldots, N$, $\Phi(i,\varphi)$ depends from values of the function $\varphi(j)$ by $j = 0, 1, \ldots, i$ only, the functionals $a(i,j,\varphi,u)$, $b(i,j,\varphi,u)$, $G(j,\varphi,u)$ depend from values of the function $\varphi(l)$ by $l = 0, 1, \ldots, j$ only and satisfy to the conditions

$$|\Phi(i,\varphi)| \le \sum_{j=0}^{i} (1 + |\varphi(j)|)\, K_0(j), \quad (5)$$

$$|a(i,j,\varphi,u)|^2 + |b(i,j,\varphi,u)|^2 \le \sum_{l=0}^{j} \left(1 + |u|^2 + |\varphi(l)|^2\right) K_1(l), \quad (6)$$

$$|\Phi(i,\varphi_1) - \Phi(i,\varphi_2)| \le \sum_{j=0}^{i} |\varphi_1(j) - \varphi_2(j)| K_0(j), \quad (7)$$

$$|\nabla\Phi(i,\varphi_1), \varphi| \le \sum_{j=0}^{i} |\varphi(j)| K_0(j), \quad (8)$$

$$|\nabla a(i,j,\varphi_1,u)\varphi|^2 + |\nabla b(i,j,\varphi_1,u)\varphi|^2 \le \sum_{l=0}^{j} |\varphi(l)|^2 K_1(l), \quad (9)$$

$$|\nabla_u a(i,j,\varphi,u)\varphi|^2 + |\nabla_u b(i,j,\varphi,u)\varphi|^2 \le C, \quad (10)$$

$$|(\nabla\Phi(i,\varphi_1) - \nabla\Phi(i,\varphi_2))\varphi|^2 \le \sum_{j=0}^{i} |\varphi_1(j) - \varphi_2(j)|^2 K_0(j), \quad (11)$$

$$|(\nabla a(i,j,\varphi_1,u_1)-\nabla a(i,j,\varphi_2,u_2))\varphi|^2+|(\nabla b(i,j,\varphi_1,u_1)-\nabla b(i,j,\varphi_2,u_2))\varphi|^2 \le \sum_{l=0}^{j}\left[|\varphi_1(l)-\varphi_2(l)|^2+|u_1-u_2|^2\right]|\varphi(l)|^2 K_1(l), \tag{12}$$

$$|\nabla_u a(i,j,\varphi,u_1)-\nabla_u a(i,j,\varphi,u_2)|^2+|\nabla_u b(i,j,\varphi,u_1)-\nabla_u b(i,j,\varphi,u_2)|^2 \le C|u_1-u_2|. \tag{13}$$

Here ∇ is Gâteaux derivative with respect to φ and ∇_u is derivative with respect to u.

For the cost functional it is assumed that

$$|F(\varphi)| \le \sum_{j=0}^{N}(1+|\varphi(j)|^2)\,K_1(j), \tag{14}$$

$$|G(i,\varphi,u)| \le \sum_{j=0}^{i}(1+|u|^2+|\varphi(j)|^2)\,K_1(j), \tag{15}$$

$$|\langle\nabla F(\varphi_1),\varphi\rangle| \le \sum_{j=0}^{N}(1+|\varphi_1(j)|)\,|\varphi(j)|^2 K_1(j), \tag{16}$$

$$|\langle\nabla G(i,\varphi_1,u),\varphi\rangle| \le \sum_{j=0}^{i}(1+|u|+|\varphi_1(j)|)\,|\varphi(j)|^2 K_1(j), \tag{17}$$

$$|\langle\nabla F(\varphi_1)-\nabla F(\varphi_2),\varphi\rangle| \le \sum_{j=0}^{N}|\varphi_1(j)-\varphi_2(j)|\,|\varphi(j)|\,K_1(j), \tag{18}$$

$$|\langle\nabla G(i,\varphi_1,u_1)-\nabla G(i,\varphi_2,u_2),\varphi\rangle| \le \sum_{j=0}^{i}(|\varphi_1(j)-\varphi_2(j)|+|u_1-u_2|)\,|\varphi(j)|K_1(j), \tag{19}$$

$$|\nabla_u G(i,\varphi,u_1)-\nabla_u G(i,\varphi,u_2)| \le C|u_1-u_2|. \tag{20}$$

Here $\max_{i\in\mathbf{Z}}|K_0(i)|<1$.

DEFINITION. *The matrix $b(i,j)$ is called the resolvent of the kernel $a(i,j)$ if the solution of the equation*

$$x(i+1)=\eta(i+1)+\sum_{j=0}^{i}a(i,j)x(j), \qquad x(0)=\eta(0),$$

can be represented in the form

$$x(i+1)=\eta(i+1)+\sum_{j=0}^{i}b(i,j)\eta(j).$$

The kernel and the resolvent are connected by the relations:

$$b(i,j)=a(i,j)+\sum_{k=j+1}^{i}b(i,k)a(k-1,j),$$

$$b(i,j)=a(i,j)+\sum_{k=j+1}^{i}a(i,k)b(k-1,j).$$

THEOREM 1. *Let the conditions (5)–(20) hold. Then the limit (1),(2) for the problem (3),(4) there exists and equals*

$$J_0(u_0) = \mathbf{E}\left[\langle F_0(N), q_{0N}\rangle + \sum_{j=0}^{N-1}\left(\langle G_0(j), q_{0j}\rangle + v'(j) g_0(j)\right)\right]. \tag{21}$$

Here $q_0(j)$ is the solution of the stochastic equation

$$q_0(i+1) = \eta_0(i+1) + \Phi_0(i+1) q_{0,i+1} + \sum_{j=0}^{i} A_0(i,j) q_{0j} + \sum_{j=0}^{i} B_0(i,j) q_{0j}\xi(j), \quad q_0(0) = 0, \tag{22}$$

where

$$\eta_0(i) = \sum_{j=0}^{i} a_0(i,j) v(j) + \sum_{j=0}^{i} b_0(i,j) v(j)\xi(j),$$

$$\begin{aligned} F_0(N) &= \nabla F(x_{0N}), & \Phi_0(i) &= \nabla\Phi(i, x_{0i}),\\ G_0(j) &= \nabla G(j, x_{0j}, u_0(j)), & g_0(j) &= \nabla_u G(j, x_{0j}, u_0(j)),\\ A_0(i,j) &= \nabla a(i,j,x_{0j}, u_0(j)), & a_0(i,j) &= \nabla_u a(i,j,x_{0j},u_0(j)),\\ B_0(i,j) &= \nabla b(i,j,x_{0j},u_0(j)), & b_0(i,j) &= \nabla_u b(i,j,x_{0j},u_0(j)). \end{aligned} \tag{23}$$

For the proving of the theorem we need the auxiliary statements.

3. AUXILIARY STATEMENTS

LEMMA 1. *Let*

$$0 \le z(0) \le z(1) \le \cdots \le z(N-1),$$
$$y(i) \ge 0, \qquad i = 1, 2, \ldots, N, \qquad y(1) \le Cz(0),$$
$$y(i+1) \le C\left[z(i) + \sum_{j=1}^{i} y(j)\right].$$

Then there exists the constant $C_1 > 0$ independent of $z(i)$, $y(i)$ and such that

$$y(i+1) \le C_1 z(i), \qquad i = 0, 1, \ldots, N-1.$$

PROOF. Using the mathematical induction method let us supposed that $y(j) \le Cz(j-1)$ for $j = 1, \ldots, i$. In particular, it holds by $i = 1$. Let us prove it for $i+1$. Since $z(j-1) \le z(i)$ by $j \le i$ then

$$y(i+1) \le C\left[z(i) + \sum_{j=1}^{i} Cz(j-1)\right] \le C\left[z(i) + \sum_{j=1}^{i} Cz(i)\right] \le C_1 z(i),$$

where $C_1 = C(1 + CN)$. The lemma is proved.

LEMMA 2. *Let $x_\varepsilon(j)$ be the solution of the equation (3) by the control $u_\varepsilon(j)$,*

$$q_\varepsilon(i) = \frac{1}{\varepsilon}\left[x_\varepsilon(i) - x_0(i)\right], \tag{24}$$

and the conditions (7)–(12) hold. Then $q_\varepsilon \in H$ uniformly for $\varepsilon \ge 0$.

PROOF. From (3), it follows

$$q_\varepsilon(i+1) = \frac{1}{\varepsilon}\left[\Phi(i+1, x_{\varepsilon,i+1}) - \Phi(i+1, x_{0,i+1})\right] + \frac{1}{\varepsilon}\sum_{j=0}^{i}\left[a(i,j,x_{\varepsilon j}, u_\varepsilon(j)) - a(i,j,x_{0j}, u_0(j))\right]$$
$$+ \frac{1}{\varepsilon}\sum_{j=0}^{i}\left[b(i,j,x_{\varepsilon j}, u_\varepsilon(j)) - b(i,j,x_{0j}, u_0(j))\right]\xi(j).$$

Let

$$\lambda_\varepsilon^\tau(i) = x_0(i) + \tau\varepsilon q_\varepsilon(i), \qquad u_\varepsilon^\tau(i) = u_0(i) + \tau\varepsilon v(i), \qquad \Phi_\varepsilon(i) = \int_0^1 \nabla\Phi(i, \lambda_\varepsilon^\tau)\, d\tau,$$

$$A_\varepsilon(i,j) = \int_0^1 \nabla a(i,j,\lambda_\varepsilon^\tau(j), u_\varepsilon(j))\, d\tau, \qquad a_\varepsilon(i,j) = \int_0^1 \nabla_u a(i,j,x_{0j}, u_\varepsilon^\tau(j))\, d\tau,$$
$$B_\varepsilon(i,j) = \int_0^1 \nabla b(i,j,\lambda_\varepsilon^\tau(j), u_\varepsilon(j))\, d\tau, \qquad b_\varepsilon(i,j) = \int_0^1 \nabla_u b(i,j,x_{0j}, u_\varepsilon^\tau(j))\, d\tau.$$

Then

$$\begin{aligned} q_\varepsilon(i+1) = \Phi_\varepsilon(i+1) q_{\varepsilon,i+1} &+ \sum_{j=0}^{i} A_\varepsilon(i,j) q_{\varepsilon j} + \sum_{j=0}^{i} B_\varepsilon(i,j) q_{\varepsilon j}\xi(j) \\ &+ \sum_{j=0}^{i} a_\varepsilon(i,j) v(j) + \sum_{j=0}^{i} b_\varepsilon(i,j) v(j)\xi(j). \end{aligned} \tag{25}$$

From (9),(25) it follows

$$|q_\varepsilon(i+1)|(1 - K_0(i+1)) \le \sum_{j=0}^{i} |q_\varepsilon(j)| K_0(j) + \sum_{j=0}^{i} |A_\varepsilon(i,j) q_{\varepsilon j}|$$
$$+ \left|\sum_{j=0}^{i} B_\varepsilon(i,j) q_{\varepsilon j}\xi(j)\right| + \sum_{j=0}^{i} |a_\varepsilon(i,j) v(j)| + \left|\sum_{j=0}^{i} b_\varepsilon(i,j) v(j)\xi(j)\right|.$$

Squaring this expression and calculating the mathematical expectation by virtue of (9),(10) we obtain

$$\begin{aligned} \mathbf{E}|q_\varepsilon(i+1)|^2 &\le C\left[\sum_{j=0}^{i} \mathbf{E}|q_\varepsilon(j)|^2 K_0(j) + \sum_{j=0}^{i}\sum_{l=0}^{j} \mathbf{E}|q_\varepsilon(l)|^2 K_1(l) + \sum_{j=0}^{i} \mathbf{E}|v(j)|^2\right] \\ &= C\left[\sum_{j=0}^{i} \mathbf{E}|q_\varepsilon(j)|^2 (K_0(j) + (i-j+1)K_1(j)) + \sum_{j=0}^{i} \mathbf{E}|v(j)|^2\right]. \end{aligned}$$

Let be $y_\varepsilon(i) = \mathbf{E}|q_\varepsilon(i)|^2$, $z(i) = \sum_{j=0}^{i} \mathbf{E}|v(j)|^2$. Then for some $C > 0$ we obtain

$$y_\varepsilon(i+1) \le C\left[z(i) + \sum_{j=1}^{i} y(j)\right]. \tag{26}$$

Using Lemma 1 and $\mathbf{E}|v(j)|^2 < \infty$ we obtain that $q_\varepsilon \in H$ uniformly for $\varepsilon \ge 0$. The lemma is proved.

LEMMA 3. *Let the conditions (5)–(13) hold. Then* $\lim_{\varepsilon\to 0} \|q_\varepsilon - q_0\|_i = 0$.

PROOF. From (25) it follows

$$q_\varepsilon(i+1) = \Phi_\varepsilon(i+1)q_{\varepsilon,i+1} - \Phi_0(i+1)q_{0,i+1} + \sum_{j=0}^{i}[A_\varepsilon(i,j)q_{\varepsilon j} - A_0(i,j)q_{0j}]$$
$$+\sum_{j=0}^{i}[B_\varepsilon(i,j)q_{\varepsilon j} - B_0(i,j)q_{0j}]\xi(j) + \sum_{j=0}^{i}[a_\varepsilon(i,j) - a_0(i,j)]v(j) + \sum_{j=0}^{i}[b_\varepsilon(i,j) - b_0(i,j)]v(j)\xi(j).$$

Therefore

$$q_\varepsilon(i+1) = \Phi_0(i+1)[q_\varepsilon - q_0]_{i+1} + (\Phi_\varepsilon(i+1) - \Phi_0(i+1))q_{\varepsilon,i+1} + \sum_{j=0}^{i}(A_0(i,j)[q_\varepsilon - q_0]_j$$
$$+ (A_\varepsilon(i,j) - A_0(i,j))q_{\varepsilon j}) + \sum_{j=0}^{i}(B_0(i,j)[q_\varepsilon - q_0]_j + (B_\varepsilon(i,j) - B_0(i,j))q_{\varepsilon j})\xi(j)$$
$$+\sum_{j=0}^{i}(a_\varepsilon(i,j) - a_0(i,j))v(j) + \sum_{j=0}^{i}(b_\varepsilon(i,j) - b_0(i,j))v(j)\xi(j).$$

Using (8),(9) analogously (26) we have

$$y_\varepsilon(i+1) \le C\left[\sum_{j=0}^{5}\alpha_i(j) + \sum_{j=0}^{i} y_\varepsilon(j)\right]. \tag{27}$$

Here

$$y_\varepsilon(i) = \mathbf{E}|q_\varepsilon(i) - q_0(i)|^2, \qquad \alpha_1(\varepsilon,i) = \mathbf{E}\left|[\Phi_\varepsilon(i) - \Phi_0(i)]q_{\varepsilon j}\right|^2,$$
$$\alpha_2(\varepsilon,i) = \mathbf{E}\left|[A_\varepsilon(i,j) - A_0(i,j)]q_{\varepsilon j}\right|^2, \qquad \alpha_3(\varepsilon,i) = \mathbf{E}\left|[B_\varepsilon(i,j) - B_0(i,j)]q_{\varepsilon j}\right|^2,$$
$$\alpha_4(\varepsilon,i) = \mathbf{E}\left|[a_\varepsilon(i,j) - a_0(i,j)]v(j)\right|^2, \qquad \alpha_5(\varepsilon,i) = \mathbf{E}\left|[b_\varepsilon(i,j) - b_0(i,j)]v(j)\right|^2.$$

Let us prove that $\lim_{\varepsilon\to 0}\alpha_1(\varepsilon,i) = 0$ uniformly for $i \in Z$. Let be $\chi_\varepsilon^K(i)$ the indicator of the set $\{\omega : |q_\varepsilon(i)| > K\}$. Represent $q_\varepsilon(i)$ in the form

$$q_\varepsilon(i) = q_\varepsilon(i)\chi_\varepsilon^K(i) + q_\varepsilon(i)(1 - \chi_\varepsilon^K(i)).$$

We obtain

$$\alpha_1(\varepsilon,i) = \mathbf{E}\left|(\Phi_\varepsilon(i) - \Phi_0(i))q_\varepsilon(i)\chi_\varepsilon^K(i) + q_\varepsilon(i)(1-\chi_\varepsilon^K(i))\right|^2$$
$$= \mathbf{E}\left|\Phi_\varepsilon(i)q_\varepsilon(i)\chi_\varepsilon^K(i) - \Phi_0(i)q_\varepsilon(i)\chi_\varepsilon^K(i) + (\Phi_\varepsilon(i) - \Phi_0(i))q_\varepsilon(i)(1-\chi_\varepsilon^K(i))\right|^2$$
$$\le 3\left[\mathbf{E}\left|\Phi_\varepsilon(i)(q_\varepsilon\chi_\varepsilon^K)_i\right|^2 + \mathbf{E}\left|\Phi_0(i)(q_\varepsilon\chi_\varepsilon^K)_i\right|^2 + \mathbf{E}\left|(\Phi_\varepsilon(i) - \Phi_0(i))\left[q_\varepsilon(1-\chi_\varepsilon^K)\right]_i\right|^2\right].$$

Using (8),(11) we obtain

$$\alpha_1(\varepsilon,i) \le C\left[\sum_{j=0}^{i}\left|\mathbf{E}(q_\varepsilon\chi_\varepsilon^K)_j\right|^2 K_0(j)\right.$$
$$\left. + \sum_{j=0}^{i}\mathbf{E}\left|x_\varepsilon(j) - x_0(j)\right|^2\left|(q_\varepsilon(1-\chi_\varepsilon^K))_j\right| K_1(j)] \le C[\left\|q_\varepsilon\chi_\varepsilon^K\right\|_N^2 + \varepsilon^2K^2\right].$$

As $q_\varepsilon \in H$ uniformly for ε, then $\lim_{K\to\infty} \|q_\varepsilon \chi_\varepsilon^K\|_N = 0$. Hence, for all $\delta > 0$, there exist K, that $\|q_\varepsilon \chi_\varepsilon^K\|_N^2 < \delta(2C)^{-1}$. Fixing this K, choose ε so that $\varepsilon^2 K^2 < \delta(2C)^{-1}$. Therefore for all $\delta > 0$ there exists $\varepsilon > 0$ that $\alpha_1(\varepsilon, i) < \delta$.

Analogously, it is proved that $\lim_{\varepsilon\to 0} \alpha_i(\varepsilon, i) = 0$, $i = 2, \dots, 5$. From (27) it follows, that for all $\delta > 0$ there exists $\varepsilon > 0$, that

$$y_\varepsilon(i+1) \le C\left[\delta + \sum_{j=1}^{i} y_\varepsilon(j)\right].$$

From the Lemma 1, we obtain

$$\|q_\varepsilon - q_0\|_i \le C_1\delta.$$

The lemma is proved.

4. LIMIT CALCULATION

From (1) it follows

$$\begin{aligned} J_\varepsilon(u_0) = \frac{1}{\varepsilon}[J(u_\varepsilon) - J(u_0)] &= \mathbf{E}\left[\frac{1}{\varepsilon}(F(x_{\varepsilon N}) - F(x_{0N}))\right. \\ &\left. +\frac{1}{\varepsilon}\sum_{j=0}^{N-1}(G(j, x_{\varepsilon j}, u_\varepsilon(j)) - G(j, x_{0j}, u_0(j)))\right] \\ &= \mathbf{E}\langle F_\varepsilon(N), q_{\varepsilon N}\rangle + \sum_{j=0}^{N-1}(\langle G_\varepsilon(j), q_{\varepsilon j}\rangle + v'(j) g_\varepsilon(j))]. \end{aligned}$$

Here

$$F_\varepsilon(N) = \int_0^1 \nabla F(N, \lambda_\varepsilon^\tau)\, d\tau, \qquad G_\varepsilon(j) = \int_0^1 \nabla G(j, \lambda_{\varepsilon j}^\tau, u_0(j))\, d\tau,$$

$$g_\varepsilon(j) = \int_0^1 \nabla_u G(j, x_{0j}, u_\varepsilon^\tau(j))\, d\tau.$$

From here it follows

$$J_\varepsilon(u_0) = \mathbf{E}\left[\langle F_0(N), q_{\varepsilon N}\rangle + \sum_{j=0}^{N-1}\langle G_0(j), q_{\varepsilon j}\rangle v'(j) g_0(j) + \sum_{j=0}^{3}\beta_j(\varepsilon)\right],$$

where

$$\beta_1(\varepsilon) = \mathbf{E}\langle F_\varepsilon(N) - F_0(N), q_{\varepsilon N}\rangle, \qquad \beta_2(\varepsilon) = \mathbf{E}\sum_{j=0}^{N-1}\langle G_\varepsilon(j) - G_0(j), q_{\varepsilon j}\rangle,$$

$$\beta_3(\varepsilon) = \mathbf{E}\sum_{j=0}^{N-1} v'(j)[g_\varepsilon(j) - g_0(j)].$$

Let us prove that $\lim_{\varepsilon\to 0}\beta_1(\varepsilon) = 0$. From (18) it follows

$$\beta_1(\varepsilon) \le \sum_{j=0}^{N}\mathbf{E}|x_\varepsilon(j) - x_0(j)|\ |q_\varepsilon(j)| K_1(j) = \varepsilon\sum_{j=0}^{N}\mathbf{E}|q_\varepsilon(j)|^2 K_1(j) \le C\varepsilon\|q_\varepsilon\|_N^2.$$

Analogously, estimations for $\beta_2(\varepsilon)$, $\beta_3(\varepsilon)$ is true. From here we have

$$\lim_{\varepsilon\to 0}|J_\varepsilon(u_0) - J_0(u_0)| = 0.$$

The theorem is proved.

5. LINEAR-SQUARE OPTIMAL CONTROL PROBLEM

Consider the linear-square optimal control problem with the motion trajectory

$$x(i+1) = \eta(i+1) + \sum_{j=0}^{i} a(i,j)x(j) + \sum_{j=0}^{i} b(i,j)u(j), \qquad x(0) = \eta(0), \tag{28}$$

and the cost functional

$$J(u) = \mathbf{E}\left[x'(N)Fx(N) + \sum_{j=0}^{N-1} u'(j)G(j)u(j)\right]. \tag{29}$$

Here $x(i) \in \mathbf{R}^n$, $u(i) \in \mathbf{R}^m$, $a(i,j)$ and $b(i,j)$ are deterministic $n \times n$ and $n \times m$-matrices, F is positive semidefinite matrix and $G(j)$, $j \in Z$, are positive definite matrices.

For the problem (28),(29) the limit (21) has a form

$$J_0(u_0) = 2\mathbf{E}\left[x_0'(N)Fq_0(N) + \sum_{j=0}^{i} u_0'(j)G(j)v(j)\right].$$

Here x_0 is the solution of equation (28) by the control u_0 and $q_0(i)$ is the solution of the equation

$$q_0(i+1) = \sum_{j=0}^{i} a(i,j)q_0(j) + \sum_{j=0}^{i} b(i,j)v(j), \qquad q_0(0) = 0.$$

In the linear case the equation $J_0(u_0) = 0$ has the unique solution u_0 [5]

$$u_0(j+1) = \alpha(j+1) + p(j+1)\psi(N-1,j,I)x_0(j+1) + \sum_{k=0}^{j} \gamma(j,k)x_0(k), \quad j = 0,1,\dots,N-2,$$

$$u_0(0) = \alpha(0) + p(0)\left(1 + \sum_{k=0}^{N-1} R(N-1,k)\right)x_0(0).$$

Here I is identical matrix,

$$p(j) = -G^{-1}(j)\psi'(N-1,j,b(\cdot,j)) \times F\left[I + \sum_{k=j}^{N-1} \psi(N-1,k,b(\cdot,k))G^{-1}(k)\psi'(N-1,k,b(\cdot,k))F\right]^{-1},$$

$$\alpha(j+1) = p(j+1)\psi(N-1,j,\beta(\cdot,j+1)) + \sum_{k=0}^{j} Q(j,k)\psi(N-1,k-1,\beta(\cdot,k)),$$

$$\gamma(j,k) = p(j+1)\psi(N-1,j,a_j(\cdot,k)) + Q(j,k)p(k)\psi(N-1,k-1,I) + \sum_{l=k+1}^{j} Q(j,l)p(l)\psi(N-1,l-1,a_{l-1}(\cdot,k)),$$

$Q(j,k)$ is the resolvent of the kernel $p(j+1)\psi(N-1,j,b_j(\cdot,k))$,

$$a_j(i,k) = a(i,k) - a(j,k), \qquad b_j(i,k) = b(i,k) - b(j,k), \qquad \beta(i,j) = \mathbf{E}_j\eta(i+1) - \eta(j).$$

REMARK 1. If process $\eta(i)$ is martingale then $\beta(i,j) = 0$, $i \geq j$, and therefore $\alpha(i) = 0$, $i \in Z$.

REMARK 2. Analogously we can obtain the optimal control for system with noice by the control. For example, consider scalar equation

$$x(i+1) = \eta(i+1) + \sum_{j=0}^{i} [\beta(i,j) + \gamma(i,j)u(j)]\xi(j), \qquad x(0) = \eta(0),$$

with the cost functional

$$J(u) = \mathbf{E}\left[x^2(N) + \lambda \sum_{i=0}^{N-1} u^2(i)\right], \qquad \lambda > 0.$$

This example was considered in [5].

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