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About one application of the general method of Lyapunov functionals construction

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SUMMARY

Construction of Lyapunov functionals are used for problems of stability and optimal control of hereditary systems which are described usually by functional-differential equations or Volterra equations and have numerous applications (*Stability of Functional Differential Equations*. Academic Press: New York, 1986; *Applied Theory of Functional Differential Equations*. Kluwer Academic Publishers: Boston, 1992; *Translations of Mathematical Monographs*, vol. 157. American Mathematical Society, Providence, RI, 1996; *Lesons sur la theorie mathematique de la lutte pour la vie*. Gauthier-Villars: Paris, 1931). The general method of Lyapunov functionals construction for stability investigation of hereditary systems was proposed and developed during last decade (*Dynamical Systems and Applications*, vol. 1. Dynamic Publishers: New York, 1994, 167; *Dynamical Systems and Applications*, vol. 4, World Scientific Series in Applicable Analysis, Singapore, 1995, 397; *Differentialniye uravneniya*, vol. 31, No. 11, 1995, 1851; *Dynamic Systems and Applications*, vol. 4, No. 2, 1995, 1999; *Theory Stochastic Processes* 1996; **2**(18), (1–2):248; *Appl. Math. Lett*. 1997; **10**(3): 111; *Funct. Differential Equations* 1997; **4**(3–4) 279; *Advances in Systems, Signals, Control and Computers*, Bajic V (ed.). IAAMSAD and SA branch of the Academy of Nonlinear Sciences, Durban, South Africa. vol. 1, 1998; 97; *J. Appl. Math. Mech*. 1999; **63**(4):537; *Stability Control Theory Appl*. 2000; **3**(1):24; *Dynamic Systems Appl*. 2000; 9(4):501; *Appl. Math. Lett*. 2002; **15**(3):355; *Math. Comput. Modelling* 2002; **36**(6):691) both for stochastic differential equations with aftereffect and stochastic difference Volterra equations. Here, some development of this method for stochastic difference Volterra equations is considered. It is shown, in particular, that for given equation with unbounded memory the proposed method of Lyapunov functionals construction allows to get a sequence of extending stability regions. Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: stochastic difference equations; delay; stability; method of Lyapunov functionals construction

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1. STATEMENT OF THE PROBLEM

Let i be a discrete time, $i \in Z_0 \cup Z$, $Z_0 = \{-h, \ldots, 0\}$, $Z = \{0, \mathbf{1}, \ldots\}$, h be a given non-negative number, process $x_i \in \mathbf{R}^n$ be a solution of the equation

$$x_{i+1} = F(i, x_{-h}, \ldots, x_i) + \sum_{j=0}^{i} G(i, j, x_{-h}, \ldots, x_j)\xi_j, \quad i \in Z,$$

$$x_i = \phi_i, \quad i \in Z_0. \tag{1}$$

Here $F : Z{*}S \Rightarrow \mathbf{R}^n$, $G : Z{*}Z{*}S \Rightarrow \mathbf{R}^n$, S is a space of sequences with elements from $\mathbf{R}^n$. It is assumed that $F(i, \ldots)$ does not depend on x_j for $j > i$, $G(i, j, \ldots)$ does not depend on x_k for $k > j$ and $F(i, 0, \ldots, 0) = 0$, $G(i, j, 0, \ldots, 0) = 0$.

Let $\{\Omega, \sigma, \mathbf{P}\}$ be a basic probability space, $f_i \in \sigma$, $i \in Z$, be a family of σ-algebras, $\xi_0, \xi_1, \ldots$ be a sequence of mutually independent random values, ξ_i be f_{i+1}-adapted and independent on f_i, $\mathbf{E}\xi_i = 0$, $\mathbf{E}\xi_i^2 = 1$, $i \in Z$.

Definition 1.1
The trivial solution of equation (1) is called p-stable, $p > 0$, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mathbf{E}|x_i|^p < \varepsilon$, $i \in Z$, if $\|\phi\|^p = \sup_{i \in Z_0} \mathbf{E}|\phi(i)|^p < \delta$. If, besides, $\lim_{i\to\infty} \mathbf{E}|x_i|^p = 0$ then the trivial solution of equation (1) is called asymptotically p-stable. In particular, if $p = 2$ then the trivial solution of equation (1) is called asymptotically mean square stable.

Theorem 1.1 (Kolmanovskii and Shaikhet [1])
Let there exists a non-negative functional $V_i = V(i, x_{-h}, \ldots, x_i)$, for which the conditions

$$\mathbf{E}V(0, x_{-h}, \ldots, x_0) \leqslant c_1 \|\phi\|^p \tag{2}$$

$$\mathbf{E}\Delta V_i \leqslant -c_2 \mathbf{E}|x_i|^p, \quad i \in Z \tag{3}$$

hold. Here $c_1 > 0$, $c_2 > 0$, $p > 0$ and

$$\Delta V_i = V(i+1, x_{-h}, \ldots, x_{i+1}) - V(i, x_{-h}, \ldots, x_i) \tag{4}$$

Then the trivial solution of equation (1) is asymptotically p-stable.

From this theorem it follows that a problem of stability investigation of some hereditary system can be reduced to the problem of construction of appropriate Lyapunov functionals. Below some formal procedure of Lyapunov functionals construction for equation type of (1) is described.

2. FORMAL PROCEDURE OF LYAPUNOV FUNCTIONALS CONSTRUCTION

The proposed procedure of Lyapunov functionals construction consists of four steps.

1. Represent the functionals F and G at the right-hand side of Equation (1) in the form

$$F(i, x_{-h}, \ldots, x_i) = F_1(i, x_{i-\tau}, \ldots, x_i) + F_2(i, x_{-h}, \ldots, x_i) + \Delta F_3(i, x_{-h}, \ldots, x_i)$$

$$F_1(i, 0, \ldots, 0) \equiv F_2(i, 0, \ldots, 0) \equiv F_3(i, 0, \ldots, 0) \equiv 0$$

$$G(i,j,x_{-h},\dots,x_j) = G_1(i,j,x_{j-\tau},\dots,x_j) + G_2(i,j,x_{-h},\dots,x_j)$$

$$G_1(i,j,0,\dots,0) \equiv G_2(i,j,0,\dots,0) \tag{5}$$

where $\tau \geqslant 0$ is a given integer, $\Delta F_3(i,x_{-h},\dots,x_i) = F_3(i+1,x_{-h},\dots,x_{i+1}) - F_3(i,x_{-h},\dots,x_i)$.

2. Consider the auxiliary difference equation

$$y_{i+1} = F_1(i, y_{i-\tau},\dots,y_i) + \sum_{j=0}^{i} G_1(i,j,y_{j-\tau},\dots,y_j)\xi_j, \quad i \in Z \tag{6}$$

and suppose that for this equation there exists a Lyapunov function $v(i, y_{i-\tau},\dots,y_i)$, which satisfies the stability conditions of Theorem 1.1.

3. Consider Lyapunov functional V_i for initial Equation (1) in the form $V_i = V_{1i} + V_{2i}$, where the main component V_{1_i} has the form

■ ■ ■

Estimate the expression $\mathbf{E}\Delta V_{1i}$.

4. Using obtained estimation for $\mathbf{E}\Delta V_{1i}$ the additional component V_2 of the functional $V_i = V_{1i} + V_{2i}$ can be chosen usually by some standard way.

This procedure is demonstrated in Reference [1] by considering a lot of concrete equations. But for equations type Volterra the parameter τ in representation (5) takes the values $\tau = 0$ and 1 only. Below procedure is used for $\tau = 2$ that allows us to extend region of stability of the equation under consideration.

3. LINEAR VOLTERRA EQUATIONS WITH CONSTANT COEFFICIENTS

Let us apply the proposed procedure for the investigation of the scalar equation

$$x_{i+1} = \sum_{l=-h}^{i} a_{i-l}x_l + \sum_{j=0}^{i} \sum_{l=-h}^{j} \sigma_{j-l}^{i-j} x_l \xi_j, \quad i \in Z$$

$$x_i = \phi_i, \quad i \in Z_0. \tag{7}$$

Here a_i and σ_j^i are some known constants.

1. Following the first step of the procedure let us represent the right-hand side of equation (7) in form (5) with

$$\tau = 2, \quad F_1(i,x_{i-2},x_{i-1},x_i) = a_0x_i + a_1x_{i-1} + a_2x_{i-2}$$

$$F_2(i,x_{-h},\dots,x_i) = \textstyle\sum_{l=-h}^{i-3} a_{i-l}x_l$$

$$F_3(i,x_{-h},\dots,x_i) = G_1(i,j,x_{j-2},x_{j-1},x_j) = 0$$

$$G_2(i,j,x_{-h},\dots,x_j) = \textstyle\sum_{l=-h}^{j} \sigma_{j-l}^{i-j}x_l, \quad j = 0,\dots,i, \ i = 0,1,\dots$$

2. In this case the auxiliary equation type of (6) is

$$y_{i+1} = a_0 y_i + a_1 y_{i-1} + a_2 y_{i-2} \tag{8}$$

Introduce into consideration the vector $y(i) = (y_{i-2}, y_{i-1}, y_i)'$ and represent equation (8) in the form

$$y(i+1) = A\,y(i), \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_2 & a_1 & a_0 \end{pmatrix} \tag{9}$$

Consider now the matrix equation

$$A'DA - D = -U, \quad U = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{10}$$

where D is a symmetric matrix. The entries d_{ij} of the matrix D are defined by the following system of the equations:

$$\begin{aligned} a_2^2 d_{33} - d_{11} &= 0 \\ a_2 d_{13} + a_1 a_2 d_{33} - d_{12} &= 0 \\ a_2 d_{23} + a_0 a_2 d_{33} - d_{13} &= 0 \\ d_{11} + 2a_1 d_{13} + a_1^2 d_{33} - d_{22} &= 0 \\ d_{12} + a_0 d_{13} + a_0 a_1 d_{33} + (a_1 - 1) d_{23} &= 0 \\ d_{22} + 2a_0 d_{23} + (a_0^2 - 1) d_{33} &= -1 \end{aligned} \tag{11}$$

Solving system (11) we obtain

$$\begin{aligned} d_{11} &= a_2^2 d_{33} \\ d_{12} &= \frac{a_2(1-a_1)(a_1 + a_0 a_2)}{1 - a_1 - a_2(a_0 + a_2)} d_{33} \\ d_{13} &= \frac{a_2(a_0 + a_1 a_2)}{1 - a_1 - a_2(a_0 + a_2)} d_{33} \\ d_{22} &= \left[a_1^2 + a_2^2 + \frac{2a_1 a_2 (a_0 + a_1 a_2)}{1 - a_1 - a_2(a_0 + a_2)} \right] d_{33} \\ d_{23} &= \frac{(a_0 + a_2)(a_1 + a_0 a_2)}{1 - a_1 - a_2(a_0 + a_2)} d_{33} \\ d_{33} &= \left[1 - a_0^2 - a_1^2 - a_2^2 - 2\,\frac{a_1 a_2 (a_0 + a_1 a_2) + a_0 (a_0 + a_2)(a_1 + a_0 a_2)}{1 - a_1 - a_2(a_0 + a_2)} \right]^{-1} \end{aligned} \tag{12}$$

Let us suppose that the solution D of matrix equation (10) is a positive semidefinite matrix with $d_{33} > 0$. In this case the function $v_i = y'(i) D y(i)$ is Lyapunov function for

Equation (8). Really,

$$\Delta v_i = y'(i+1)Dy(i+1) - y'(i)Dy(i)$$
$$= y'(i)[A'DA - D]y(i) = -y'(i)Uy(i) = -y_i^2$$

3. Following the third step of the procedure the main part V_{1i} of Lyapunov functional $V_i = V_{1i} + V_{2i}$ must be chosen in the form $V_{1i} = x'(i)Dx(i)$ where $x(i) = (x_{i-2}, x_{i-1}, x_i)'$.

Represent equation (1) in the form

$$x(i+1) = Ax(i) + B(i) \tag{13}$$

where the matrix A is defined in (9) and $B(i) = (0, 0, b_i)'$,

$$b_i = \sum_{l=-h}^{i-3} a_{i-l}x_l + \sum_{j=0}^{i} \sum_{l=-h}^{j} \sigma_{j-l}^{i-j} x_l \xi_j \tag{14}$$

Calculating $\mathbf{E}\Delta V_{1i}$, by virtue of Equation (13) we obtain

$$\mathbf{E}\Delta V_{1i} = \mathbf{E}[x'(i+1)Dx(i+1) - x'(i)Dx(i)]$$
$$= \mathbf{E}[(Ax(i) + B(i))'D(Ax(i) + B(i)) - x'(i)Dx(i)]$$
$$= \mathbf{E}[-x^2 + 2B'(i)DAx(i) + B'(i)DB(i)]$$

Using the second and the third equations of system (11) it is easy to get

$$B'(i)DB(i) = d_{33}b_i^2$$

$$B'(i)DAx(i) = \left(\frac{d_{13}}{a_2}x_i + \frac{d_{12}}{a_2}x_{i-1} + a_2 d_{33} x_{i-2}\right) b_i$$

From (12) it follows that the expressions d_{12}/a_2 and d_{13}/a_2 are definite for $a_2 = 0$ too.

As a result, using (14) we obtain

$$\mathbf{E}\Delta V_{1i} = \mathbf{E}\left[-x_i^2 + d_{33}b_i^2 + 2\left(\frac{d_{13}}{a_2}x_i + \frac{d_{12}}{a_2}x_{i-1} + a_2 d_{33}x_{i-2}\right)b_i\right]$$
$$= -\mathbf{E}x_i^2 + d_{33}\mathbf{E}b_i^2 + 2\frac{d_{13}}{a_2}\mathbf{E}x_i b_i + 2\frac{d_{12}}{a_2}\mathbf{E}x_{i-1}b_i + 2a_2 d_{33}\mathbf{E}x_{i-2}b_i = -\mathbf{E}x_i^2 + \sum_{k=1}^{9} I_k$$

where

$$I_1 = d_{33}\mathbf{E}\left(\sum_{l=-h}^{i-3} a_{i-l}x_l\right)^2, \quad I_2 = d_{33}\mathbf{E}\left(\sum_{j=0}^{i}\sum_{l=-h}^{j} \sigma_{j-l}^{i-j}x_l\xi_j\right)^2$$

$$I_3 = 2d_{33}\mathbf{E}\sum_{k=-h}^{i-3} a_{i-k}x_k \sum_{j=0}^{i}\sum_{l=-h}^{j} \sigma_{j-l}^{i-j}x_l\xi_j$$

$$I_4 = 2\frac{d_{13}}{a_2}\mathbf{E}x_i \sum_{l=-h}^{i-3} a_{i-l}x_l, \quad I_5 = 2\frac{d_{13}}{a_2}\mathbf{E}x_i \sum_{j=0}^{i}\sum_{l=-h}^{j} \sigma_{j-l}^{i-j}x_l\xi_j$$

$$I_6 = 2\frac{d_{12}}{a_2}\mathbf{E}x_{i-1}\sum_{l=-h}^{i-3} a_{i-l}x_l, \quad I_7 = 2\frac{d_{12}}{a_2}\mathbf{E}x_{i-1}\sum_{j=0}^{i}\sum_{l=-h}^{j}\sigma_{j-l}^{i-j}x_l\xi_j$$

$$I_8 = 2a_2d_{33}\mathbf{E}x_{i-2}\sum_{l=-h}^{i-3} a_{i-l}x_l, \quad I_9 = 2a_2d_{33}\mathbf{E}x_{i-2}\sum_{j=0}^{i}\sum_{l=-h}^{j}\sigma_{j-l}^{i-j}x_l\xi_j$$

Let us estimate the summands I_k, $k = 1, \ldots, 9$, using the following notation:

$$k_m = \max(k, 0), \quad \alpha_3 = \sum_{l=3}^{\infty}|a_l|$$

$$S_0 = \sum_{p=0}^{\infty}\left(\sum_{l=0}^{\infty}|\sigma_l^p|\right)^2, \quad S_k = \sum_{i=k}^{\infty}\sum_{j=0}^{\infty}|\sigma_j^i|, \quad k = 1, 2, 3, 4 \tag{15}$$

For I_1, I_2 we have

$$I_1 \leqslant d_{33}\sum_{l=-h}^{i-3}|a_{i-l}|\sum_{k=-h}^{i-3}|a_{i-k}|\mathbf{E}x_k^2 \leqslant \alpha_3 d_{33}\sum_{k=-h}^{i-3}|a_{i-k}|\mathbf{E}x_k^2$$

$$I_2 = d_{33}\sum_{j=0}^{i}\mathbf{E}\left(\sum_{l=-h}^{j}\sigma_{j-l}^{i-j}x_l\right)^2 \leqslant d_{33}\sum_{j=0}^{i}\sum_{l=-h}^{j}|\sigma_{j-l}^{i-j}|\sum_{k=-h}^{j}|\sigma_{j-k}^{i-j}|\mathbf{E}x_k^2$$

$$\leqslant d_{33}\sum_{k=-h}^{i}\sum_{j=k_m}^{i}|\sigma_{j-k}^{i-j}|\sum_{l=0}^{\infty}|\sigma_l^{i-j}|\mathbf{E}x_k^2$$

$$= d_{33}\Bigg(|\sigma_0^0|\sum_{l=0}^{\infty}|\sigma_l^0|\mathbf{E}x_i^2 + \sum_{p=0}^{1}|\sigma_{1-p}^p|\sum_{l=0}^{\infty}|\sigma_l^p|\mathbf{E}x_{i-1}^2$$

$$+ \sum_{p=0}^{2}|\sigma_{2-p}^p|\sum_{l=0}^{\infty}|\sigma_l^p|\mathbf{E}x_{i-2}^2 + \sum_{k=-h}^{i-3}\sum_{p=0}^{i-k_m}|\sigma_{i-p-k}^p|\sum_{l=0}^{\infty}|\sigma_l^p|\mathbf{E}x_k^2\Bigg)$$

Using properties of conditional expectation for I_3 we obtain

$$|I_3| = 2d_{33}\left|\mathbf{E}\sum_{k=-h}^{i-3}a_{i-k}x_k\sum_{j=0}^{i}\sum_{l=-h}^{j}\sigma_{j-l}^{i-j}x_l\xi_j\right|$$

$$= 2d_{33}\left|\mathbf{E}\sum_{k=-h}^{i-3}a_{i-k}x_k\sum_{j=0}^{i-4}\sum_{l=-h}^{j}\sigma_{j-l}^{i-j}x_l\xi_j\right|$$

$$= 2d_{33}\left|\mathbf{E}\sum_{j=0}^{i-4}\left(\sum_{k=-h}^{j}a_{i-k}x_k + \sum_{k=j+1}^{i-3}a_{i-k}x_k\right)\sum_{l=-h}^{j}\sigma_{j-l}^{i-j}x_l\xi_j\right|$$

$$
\begin{aligned}
&= 2d_{33}\left|\mathbf{E}\sum_{j=0}^{i-4}\sum_{k=j+1}^{i-3} a_{i-k}x_k \sum_{l=-h}^{j}\sigma_{j-l}^{i-j}x_l\xi_j\right| \\
&\leqslant d_{33}\sum_{j=0}^{i-4}\sum_{l=-h}^{j}|\sigma_{j-l}^{i-j}|\sum_{k=j+1}^{i-3}|a_{i-k}|(\mathbf{E}x_l^2+\mathbf{E}x_k^2) \\
&= d_{33}\sum_{j=0}^{i-4}\sum_{l=-h}^{j}|\sigma_{j-l}^{i-j}|\sum_{k=j+1}^{i-3}|a_{i-k}|\mathbf{E}x_l^2 + d_{33}\sum_{j=0}^{i-4}\sum_{l=-h}^{j}|\sigma_{j-l}^{i-j}|\sum_{k=j+1}^{i-3}|a_{i-k}|\mathbf{E}x_k^2 \\
&\leqslant \alpha_3 d_{33}\sum_{k=-h}^{i-4}\sum_{j=k_m}^{i-4}|\sigma_{j-k}^{i-j}|\mathbf{E}x_k^2 + d_{33}\sum_{k=1}^{i-3}\sum_{j=0}^{i-4}\sum_{l=-h}^{j}|\sigma_{j-l}^{i-j}||a_{i-k}|\mathbf{E}x_k^2 \\
&\leqslant \alpha_3 d_{33}\sum_{k=-h}^{i-4}\sum_{p=4}^{i-k_m}|\sigma_{i-p-k}^{p}|\mathbf{E}x_k^2 + S_4 d_{33}\sum_{k=1}^{i-3}|a_{i-k}|\mathbf{E}x_k^2
\end{aligned}
$$

Similar for the summands $I_4 - I_9$ we get

$$
|I_4| \leqslant \left|\frac{d_{13}}{a_2}\right|\sum_{l=-h}^{i-3}|a_{i-l}|(\mathbf{E}x_i^2+\mathbf{E}x_l^2) \leqslant \left|\frac{d_{13}}{a_2}\right|\left(\alpha_3\mathbf{E}x_i^2 + \sum_{k=-h}^{i-3}|a_{i-k}|\mathbf{E}x_k^2\right)
$$

$$
\begin{aligned}
|I_5| &\leqslant \left|2\frac{d_{13}}{a_2}\mathbf{E}x_i\sum_{j=0}^{i-1}\sum_{l=-h}^{j}\sigma_{j-l}^{i-j}x_l\xi_j\right| \leqslant \left|\frac{d_{13}}{a_2}\right|\sum_{j=0}^{i-1}\sum_{l=-h}^{j}|\sigma_{j-l}^{i-j}|(\mathbf{E}x_i^2+\mathbf{E}x_l^2) \\
&\leqslant \left|\frac{d_{13}}{a_2}\right|\left(S_1\mathbf{E}x_i^2 + |\sigma_0^1|\mathbf{E}x_{i-1}^2 + \sum_{p=0}^{2}|\sigma_{2-p}^{p}|\mathbf{E}x_{i-2}^2 + \sum_{k=-h}^{i-3}\sum_{p=4}^{i-k_m}|\sigma_{i-p-k}^{p}|\mathbf{E}x_k^2\right)
\end{aligned}
$$

$$
|I_6| \leqslant \left|\frac{d_{12}}{a_2}\right|\sum_{l=-h}^{i-3}|a_{i-l}|(\mathbf{E}x_{i-1}^2+\mathbf{E}x_l^2) \leqslant \left|\frac{d_{12}}{a_2}\right|\left(\alpha_3\mathbf{E}x_{i-1}^2 + \sum_{k=-h}^{i-3}|a_{i-k}|\mathbf{E}x_k^2\right)
$$

$$
\begin{aligned}
|I_7| &= \left|2\frac{d_{12}}{a_2}\mathbf{E}x_{i-1}\sum_{j=0}^{i-2}\sum_{l=-h}^{j}\sigma_{j-l}^{i-j}x_l\xi_j\right| \leqslant \left|\frac{d_{12}}{a_2}\right|\sum_{j=0}^{i-2}\sum_{l=-h}^{j}|\sigma_{j-l}^{i-j}|(\mathbf{E}x_{i-1}^2+\mathbf{E}x_l^2) \\
&\leqslant \left|\frac{d_{12}}{a_2}\right|\left(S_2\mathbf{E}x_{i-1}^2 + |\sigma_0^2|\mathbf{E}x_{i-2}^2 + \sum_{k=-h}^{i-3}\sum_{p=2}^{i-k_m}|\sigma_{i-p-k}^{p}|\mathbf{E}x_k^2\right)
\end{aligned}
$$

$$
|I_8| \leqslant |a_2|d_{33}\sum_{l=-h}^{i-3}|a_{i-l}|(\mathbf{E}x_{i-2}^2+\mathbf{E}x_l^2) \leqslant |a_2|d_{33}\left(\alpha_3\mathbf{E}x_{i-2}^2 + \sum_{k=-h}^{i-3}|a_{i-k}|\mathbf{E}x_k^2\right)
$$

$$
\begin{aligned}
|I_9| &= \left|2a_2d_{33}\mathbf{E}x_{i-2}\sum_{j=0}^{i-3}\sum_{l=-h}^{j}\sigma_{j-l}^{i-j}x_l\xi_j\right| \leqslant |a_2|d_{33}\sum_{j=0}^{i-3}\sum_{l=-h}^{j}|\sigma_{j-l}^{i-j}|(\mathbf{E}x_{i-2}^2+\mathbf{E}x_l^2) \\
&\leqslant |a_2|d_{33}\left(S_3\mathbf{E}x_{i-2}^2 + \sum_{k=-h}^{i-3}\sum_{p=3}^{i-k_m}|\sigma_{i-p-k}^{p}|\mathbf{E}x_k^2\right)
\end{aligned}
$$

As a result we obtain

$$\mathbf{E}\Delta V_{1i} \leqslant (\gamma_0 - 1)\mathbf{E}x_i^2 + \gamma_1 \mathbf{E}x_{i-1}^2 + \gamma_2 \mathbf{E}x_{i-2}^2 + \sum_{k=-h}^{i-3} P_{ik}\mathbf{E}x_k^2 \tag{16}$$

where

$$\begin{aligned}
\gamma_0 &= d_{33}|\sigma_0^0| \sum_{l=0}^{\infty} |\sigma_l^0| + \left|\frac{d_{13}}{a_2}\right|(S_1 + \alpha_3) \\
\gamma_1 &= d_{33} \sum_{p=0}^{1} |\sigma_{1-p}^p| \sum_{l=0}^{\infty} |\sigma_l^p| + \left|\frac{d_{13}}{a_2}\right| |\sigma_0^1| + \left|\frac{d_{12}}{a_2}\right|(S_2 + \alpha_3) \\
\gamma_2 &= d_{33} \sum_{p=0}^{2} |\sigma_{2-p}^p| \sum_{l=0}^{\infty} |\sigma_l^p| + \left|\frac{d_{13}}{a_2}\right| \sum_{p=1}^{2} |\sigma_{2-p}^p| \\
&\quad + \left|\frac{d_{12}}{a_2}\right| |\sigma_0^2| + |a_2| d_{33}(S_3 + \alpha_3)
\end{aligned} \tag{17}$$

$$\begin{aligned}
P_{ik} &= d_{33}\left(S_4|a_{i-k}| + \sum_{p=0}^{i-k_m} |\sigma_{i-p-k}^p| \sum_{l=0}^{\infty} |\sigma_l^p|\right) \\
&\quad + \left|\frac{d_{13}}{a_2}\right|\left(|a_{i-k}| + \sum_{p=1}^{i-k_m} |\sigma_{i-p-k}^p|\right) + \left|\frac{d_{12}}{a_2}\right|\left(|a_{i-k}| + \sum_{p=2}^{i-k_m} |\sigma_{i-p-k}^p|\right) \\
&\quad + |a_2| d_{33}\left(|a_{i-k}| + \sum_{p=3}^{i-k_m} |\sigma_{i-p-k}^p|\right) + \alpha_3 d_{33}\left(|a_{i-k}| + \sum_{p=4}^{i-k_m} |\sigma_{i-p-k}^p|\right)
\end{aligned} \tag{18}$$

4. Following the fourth step of the procedure of Lyapunov functionals construction choose now the functional V_{2i} in the form

$$V_{2i} = (\gamma_1 + \gamma_2 + \mu)x_{i-1}^2 + (\gamma_2 + \mu)x_{i-2}^2 + \sum_{k=-h}^{i-3} x_k^2 \sum_{j=0}^{\infty} P_{j+k,k}$$

where numbers γ_1, γ_2 are defined by (17) and a positive number μ will be chosen below. Calculating ΔV_{2i} we obtain

$$\Delta V_{2i} = (\gamma_1 + \gamma_2 + \mu)x_i^2 - \gamma_1 x_{i-1}^2 - \left(\gamma_2 + \mu - \sum_{j=3}^{\infty} P_{j+i-2,i-2}\right)x_{i-2}^2 - \sum_{k=-h}^{i-3} P_{ik}x_k^2$$

Using (16), for the functional $V_i = V_{1i} + V_{2i}$ we get

$$\mathbf{E}\Delta V_i \leqslant (\gamma_0 + \gamma_1 + \gamma_2 + \mu - 1)\mathbf{E}x_i^2 + \left(\sum_{j=3}^{\infty} P_{j+i-2,i-2} - \mu\right)\mathbf{E}x_{i-2}^2$$

Putting now

$$\mu = \sup_{i \in Z} \sum_{j=3}^{\infty} P_{j+i-2,i-2} \tag{19}$$

as a result we obtain

$$\mathbf{E}\Delta V_i \leqslant (\gamma_0 + \gamma_1 + \gamma_2 + \mu - 1)\mathbf{E}x_i^2$$

Thus, if the inequality

$$\gamma_0 + \gamma_1 + \gamma_2 + \mu < 1 \tag{20}$$

holds then the constructed functional V_i satisfies the conditions of Theorem 1.1 and therefore the trivial solution of equation (7) is asymptotically mean square stable.

Let us transform condition (20) to more obvious form. For this aim estimate μ. From (15) it follows that $j + k - k_m \leqslant j$. Therefore, using (15) and (18), we have

$$\begin{aligned}\sum_{j=3}^{\infty} P_{j+i-2,i-2} \leqslant & \sum_{j=3}^{\infty}\left[d_{33}\left(S_4|a_j| + \sum_{p=0}^{j}|\sigma_{j-p}^p|\sum_{l=0}^{\infty}|\sigma_l^p|\right)\right.\\ & + \left|\frac{d_{13}}{a_2}\right|\left(|a_j| + \sum_{p=1}^{j}|\sigma_{j-p}^p|\right) + \left|\frac{d_{12}}{a_2}\right|\left(|a_j| + \sum_{p=2}^{j}|\sigma_{j-p}^p|\right)\\ & \left. + |a_2|d_{33}\left(|a_j| + \sum_{p=3}^{j}|\sigma_{j-p}^p|\right) + \alpha_3 d_{33}\left(|a_j| + \sum_{p=4}^{j}|\sigma_{j-p}^p|\right)\right]\\ = & \,\alpha_3\left[d_{33}(S_4 + |a_2| + \alpha_3) + \frac{|d_{12}| + |d_{13}|}{|a_2|}\right]\\ & + d_{33}\sum_{j=3}^{\infty}\sum_{p=0}^{j}|\sigma_{j-p}^p|\sum_{l=0}^{\infty}|\sigma_l^p| + \left|\frac{d_{13}}{a_2}\right|\sum_{j=3}^{\infty}\sum_{p=1}^{j}|\sigma_{j-p}^p| + \left|\frac{d_{12}}{a_2}\right|\sum_{j=3}^{\infty}\sum_{p=2}^{j}|\sigma_{j-p}^p|\\ & + |a_2|d_{33}\sum_{j=3}^{\infty}\sum_{p=2}^{j}|\sigma_{j-p}^p| + \alpha_3 d_{33}\sum_{j=3}^{\infty}\sum_{p=2}^{j}|\sigma_{j-p}^p|\end{aligned}$$

From here and (17) it follows that

$$\begin{aligned}\gamma_0 + \gamma_1 + \gamma_2 + \mu \leqslant & \, d_{33}|\sigma_0^0|\sum_{l=0}^{\infty}|\sigma_l^0| + \left|\frac{d_{13}}{a_2}\right|(S_1 + \alpha_3)\\ & + d_{33}\sum_{p=0}^{1}|\sigma_{1-p}^p|\sum_{l=0}^{\infty}|\sigma_l^p| + \left|\frac{d_{13}}{a_2}\right||\sigma_0^1| + \left|\frac{d_{12}}{a_2}\right|(S_2 + \alpha_3)\\ & + d_{33}\sum_{p=0}^{2}|\sigma_{2-p}^p|\sum_{l=0}^{\infty}|\sigma_l^p| + \left|\frac{d_{13}}{a_2}\right|\sum_{p=1}^{2}|\sigma_{2-p}^p| + \left|\frac{d_{12}}{a_2}\right||\sigma_0^2| + |a_2|d_{33}(S_3 + \alpha_3)\\ & + \alpha_3\left[d_{33}(S_4 + |a_2| + \alpha_3) + \frac{|d_{12}| + |d_{13}|}{|a_2|}\right]\\ & + d_{33}\sum_{j=3}^{\infty}\sum_{p=0}^{j}|\sigma_{j-p}^p|\sum_{l=0}^{\infty}|\sigma_l^p| + \left|\frac{d_{13}}{a_2}\right|\sum_{j=3}^{\infty}\sum_{p=1}^{j}|\sigma_{j-p}^p|\end{aligned}$$

$$+ \left|\frac{d_{12}}{a_2}\right| \sum_{j=3}^{\infty} \sum_{p=2}^{j} |\sigma_{j-p}^p| + |a_2| d_{33} \sum_{j=3}^{\infty} \sum_{p=3}^{j} |\sigma_{j-p}^p|$$

$$+ \alpha_3 d_{33} \sum_{j=4}^{\infty} \sum_{p=4}^{j} |\sigma_{j-p}^p|$$

$$= \left|\frac{d_{13}}{a_2}\right| (S_1 + \alpha_3) + \left|\frac{d_{12}}{a_2}\right| (S_2 + \alpha_3) + |a_2| d_{33} (S_3 + \alpha_3)$$

$$+ \alpha_3 \left[d_{33}(S_4 + |a_2| + \alpha_3) + \frac{|d_{12}| + |d_{13}|}{|a_2|} \right]$$

$$+ d_{33} \sum_{j=0}^{\infty} \sum_{p=0}^{j} |\sigma_{j-p}^p| \sum_{l=0}^{\infty} |\sigma_l^p| + \left|\frac{d_{13}}{a_2}\right| \sum_{j=1}^{\infty} \sum_{p=1}^{j} |\sigma_{j-p}^p|$$

$$+ \left|\frac{d_{12}}{a_2}\right| \sum_{j=2}^{\infty} \sum_{p=2}^{j} |\sigma_{j-p}^p| + |a_2| d_{33} \sum_{j=3}^{\infty} \sum_{p=3}^{j} |\sigma_{j-p}^p|$$

$$+ \alpha_3 d_{33} \sum_{j=4}^{\infty} \sum_{p=4}^{j} |\sigma_{j-p}^p|$$

Using notation (15) we have

$$\sum_{j=0}^{\infty} \sum_{p=0}^{j} |\sigma_{j-p}^p| \sum_{l=0}^{\infty} |\sigma_l^p| = \sum_{p=0}^{\infty} \sum_{j=p}^{\infty} |\sigma_{j-p}^p| \sum_{l=0}^{\infty} |\sigma_l^p| = \sum_{p=0}^{\infty} \left(\sum_{l=0}^{\infty} |\sigma_l^p| \right)^2 = S_0$$

$$\sum_{j=k}^{\infty} \sum_{p=k}^{j} |\sigma_{j-p}^p| = \sum_{p=k}^{\infty} \sum_{j=p}^{\infty} |\sigma_{j-p}^p| = \sum_{p=k}^{\infty} \sum_{l}^{\infty} |\sigma_{l=0}^p| = S_k, \quad k = 1, 2, 3, 4$$

Therefore, we obtain

$$\gamma_0 + \gamma_1 + \gamma_2 + \mu \leqslant \left|\frac{d_{13}}{a_2}\right| (S_1 + \alpha_3) + \left|\frac{d_{12}}{a_2}\right| (S_2 + \alpha_3) + |a_2| d_{33} (S_3 + \alpha_3)$$

$$+ \alpha_3 \left[d_{33}(S_4 + |a_2| + \alpha_3) + \frac{|d_{12}| + |d_{13}|}{|a_2|} \right]$$

$$+ d_{33} S_0 + \left|\frac{d_{13}}{a_2}\right| S_1 + \left|\frac{d_{12}}{a_2}\right| S_2 + |a_2| d_{33} S_3 + \alpha_3 d_{33} S_4$$

$$= 2\left|\frac{d_{13}}{a_2}\right| (S_1 + \alpha_3) + 2\left|\frac{d_{12}}{a_2}\right| (S_2 + \alpha_3)$$

$$+ d_{33}[S_0 + 2|a_2|(S_3 + \alpha_3) + 2\alpha_3 S_4 + \alpha_3^2]$$

Using representation (12) for d_{12}, d_{13}, we have

$$\gamma_0 + \gamma_1 + \gamma_2 + \mu \leqslant 2d_{33} \frac{|a_0 + a_1 a_2|(S_1 + \alpha_3) + |(1 - a_1)(a_1 + a_0 a_2)|(S_2 + \alpha_3)}{|1 - a_1 - a_2(a_0 + a_2)|}$$
$$+ \ d_{33}[S_0 + 2|a_2|(S_3 + \alpha_3) + 2\alpha_3 S_4 + \alpha_3^2]$$

Thus, using (12), we obtain that condition (20) follows from inequality:

$$2\frac{|a_0 + a_1 a_2|(S_1 + \alpha_3) + |(1 - a_1)(a_1 + a_0 a_2)|(S_2 + \alpha_3)}{|1 - a_1 - a_2(a_0 + a_2)|}$$
$$+ \ S_0 + 2|a_2|(S_3 + \alpha_3) + 2\alpha_3 S_4 + \alpha_3^2 < d_{33}^{-1}$$
$$= \ 1 - a_0^2 - a_1^2 - a_2^2 - 2a_0 a_1 a_2 - \frac{2(a_0 + a_2)(a_0 + a_1 a_2)(a_1 + a_0 a_2)}{1 - a_1 - a_0 a_2 - a_2^2} \qquad (21)$$

So, the following theorem is proven.

Theorem 3.1
If the matrix D with the entries defined by (12) is a positive semidefinite one and the inequality (21) holds then the trivial solution of equation (7) is asymptotically mean square stable.

4. EXAMPLES

Example 4.1
Consider the scalar equation

$$x_{i+1} = \sum_{l=-h}^{i} a_{i-l} x_l + \sigma x_{i-m} \xi_i, \quad m \geqslant 0 \qquad (22)$$

From (15) for Equation (22) it follows that:

$$S_0 = \sigma^2, \quad S_k = 0, \quad k > 0$$

Then condition (21) takes the form

$$\alpha_3^2 + 2\alpha_3 \left[|a_2| + \frac{|a_0 + a_1 a_2| + |(1 - a_1)(a_1 + a_0 a_2)|}{|1 - a_1 - a_2(a_0 + a_2)|} \right] + \sigma^2 < d_{33}^{-1} \qquad (23)$$

In particular, for the equation

$$x_{i+1} = a_0 x_i + a_1 x_{i-1} + a_2 x_{i-2} + \sigma x_{i-m} \xi_i \qquad (24)$$

we have $\alpha_3 = 0$ and from condition (23) it follows:

σ^2 [redacted] $d_{33} < 1$ (25)

As it is shown in Reference [2] if the matrix D with the entries defined by (12) is a positive semi-definite one with $d_{33} > 0$ then inequality (25) is the necessary and sufficient condition for asymptotic mean square stability of the trivial solution of equation (24).

Example 4.2
Consider the scalar equation

$$x_{i+1} = ax_i + \sum_{j=1}^{i+h} b^j x_{i-j} + \sigma x_{i-m}\xi_i$$

$$|b| < 1, \quad m \geqslant 0 \tag{26}$$

Using the procedure of Lyapunov functional construction described above for $\tau = 0$ the following condition for asymptotic mean square stability of the trivial solution of equation (26) was obtained [1]

$$|a| + \frac{|b|}{1-|b|} < \sqrt{1-\sigma^2} \tag{27}$$

Using this procedure for $\tau = 1$ in Reference [1] was obtained another stability condition, which can be written in the form

$$|a| + b < 1$$

$$\frac{b^2}{1-|b|}\left[\frac{|b|(2-|b|)}{1-|b|} + \frac{2|a|}{1-b}\right] + \sigma^2 < 1 - b^2 - a^2\frac{1+b}{1-b} \tag{28}$$

From conditions (12), (15), (21), which were obtained using the procedure for $\tau = 2$, it follows that if the symmetric matrix D with the entries:

$$d_{11} = b^4 d_{33}, \quad d_{12} = \frac{b^3(1-b)(1+ab)}{1-b-b^2(a+b^2)} d_{33}$$

$$d_{13} = \frac{b^2(a+b^3)}{1-b-b^2(a+b^2)} d_{33}, \quad d_{23} = \frac{b(a+b^2)(1+ab)}{1-b-b^2(a+b^2)} d_{33}$$

$$d_{22} = b^2\left[1 + b^2 + \frac{2b(a+b^3)}{1-b-b^2(a+b^2)}\right] d_{33}$$

$$d_{33} = \left[1 - a^2 - b^2 - b^4 - 2b\frac{b^2(a+b^3) + a(a+b^2)(1+ab)}{1-b-b^2(a+b^2)}\right]^{-1}$$

is a positive definite one and the inequality

$$\frac{|b|^6}{(1-|b|)^2} + \frac{2|b|^3}{1-|b|}\left[b^2 + \frac{|a+b^3| + (1-b)|b(1+ab)|}{|1-b-b^2(a+b^2)|}\right] + \sigma^2$$
$$< 1 - a^2 - b^2 - b^4 - 2b\frac{b^2(a+b^3) + a(a+b^2)(1+ab)}{1-b-b^2(a+b^2)} \tag{29}$$

holds then the trivial solution of equation (25) is asymptotically mean square stable.

In Figure 1, stability regions are shown, given by conditions (27) (the curve number 1), (28) (the curve number 2) and (29) (the curve number 3) for $\sigma^2 = 0$. In Figure 2 we can see the similar picture for $\sigma^2 = 0.3$ and in Figure 3 the similar picture is shown (in doubled scale) for $\sigma^2 = 0.7$.

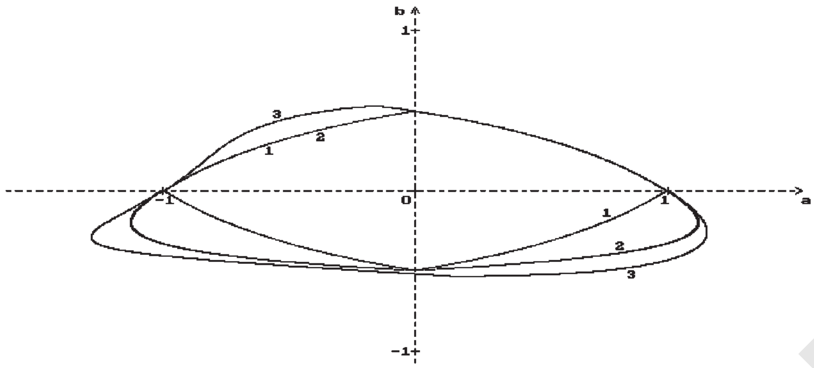

Figure 1.

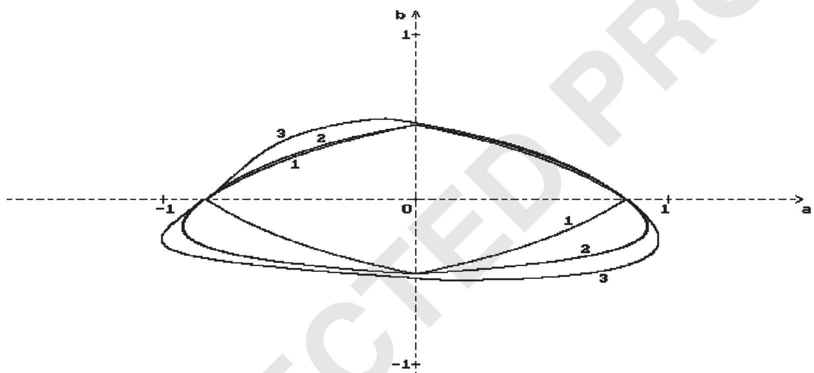

Figure 2.

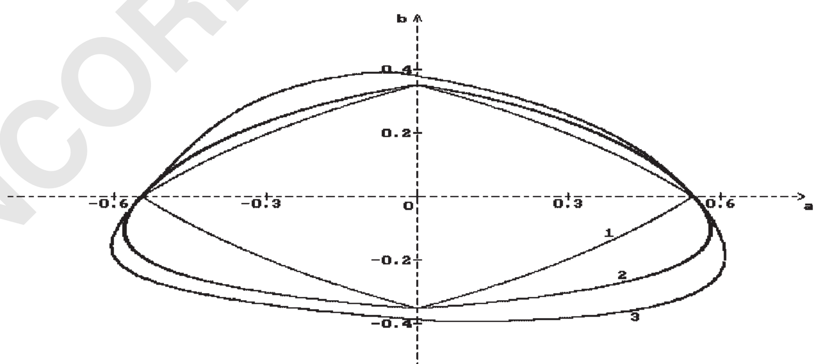

Figure 3.

Remark 4.1

As it is shown in Figure 1 (and naturally it can be shown analytically) if $\sigma = 0$ then for $b \geqslant 0$ condition (27) coincides with condition (28) and for $a \geqslant 0$, $b \geqslant 0$ condition (29) coincides with conditions (27) and (28).

Remark 4.2

It is easy to see that stability region Q_τ obtained for equation (26) expands if τ increases, i.e., $Q_0 \subset Q_1 \subset Q_2$. So, if we want to get greater stability region we can use the procedure of Lyapunov functionals construction for $\tau = 3, \tau = 4$, etc.

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