Portfolio Optimization Under Transaction Costs

Gennady Shaikhet

Portfolio Optimization Under Transaction Costs

Research thesis
Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science
in Statistics

Gennady Shaikhet

Submitted to the Senate of the Technion
Israel Institute of Technology

Nis 5763 Haifa April 2003

The work described herein was supervised by Dr. Gady Zohar
in the department of Industrial Engineering and Management.

I would like to express my gratitude to Dr. Gady Zohar
for his devoted and very professional guidance.

The generous financial help of Technion is gratefully acknowledged.

Also I thank the Wolf fund and Otto Shwartz fund for their financial support.

Finally, I thank my family, and especially my wife Alina.

Contents

List of Figures

Abstract

Over the last decades, the use of mathematics and especially stochastic analysis, has become very popular in solving different kinds of problems in finance. One of the most interesting and important is the portfolio optimization problem. Consider an investor who wants to play on a security market and who has many different opportunities of investment: risk-free bond and many kinds of stocks. Assume that the investor starts with some initial sum of money and his aim is to find the best way of investment in order to increase his wealth.

The best way to analyze such a problem is via continuous-time model. The first who pioneered such models was Merton. The main feature is that in these models the prices of the securities in the market are modelled as stochastic processes running from the beginning to some terminal (possibly infinite) time. All the information about the prices is available at each time on $[0, T]$ and it is possible for investor to trade at any moment.

We consider a financial market with two assets: a risk free asset, and a stock that behaves like an Ito process. At any instance the investor may rebalance his portfolio by moving capital from stocks to the risk free asset and vice versa in order to maximize an expected utility at some terminal time T.

For common utility functions (power, log), when there are no transaction costs and also when transaction costs are proportional to the amount traded, the solution is well known and the trading strategy is continuous in time.

But everything changes dramatically when the investor has to pay some constant fixed transaction cost for every intervention. Any trading strategy, which is continuous in time, clearly leads to bankruptcy and the optimal policy must become trading at a carefully chosen discrete sequence of instances. With fixed transaction costs the optimization problem becomes much harder to solve.

The first time such model was introduced in 1988 and the possible way of finding the optimal strategy was via so called *impulse control*. The system of quasi-variational inequalities was built and by solving this system one can obtain the optimal strategy as well as the maximized expected utility. The main problem lies in solving such inequalities. They are so difficult that it has been said that "every explicit solution is a triumph over nature".

In the present work we suggest a method to approximate the optimal policy for this problem when transaction costs are small. We also discuss the possible errors of the approximation and analyze some features of the optimal strategies.

In addition to maximizing expected utilities, we also address the problem of maximizing the probability of achieving a pre-determined goal at some terminal time. We present here some simple combinatorial solution for the no-transactions case and then we apply the approximation method for the case when fixed transactions are present.

We also use simulations in order to compare the performance of our optimal policy versus some other non-trivial policies.

List of notations and abbreviations

R^n - n-dimensional Euclidean space

L - the second order partial differential operator which coincides with the generator of Ito diffusion

W_t - standard Brownian motion

$a.s.$ - almost surely

Q^y - the probability law of the process $Y(t)$ starting at $Y(0) = y$ (Chapter 2)

HJB - the Hamilton - Jacobi - Bellman equation

$\mathcal{F}_t$ - the σ-algebra generated by $\{W_s\,; s \leq t\}$

$\mathcal{F}_\tau$ - the σ-algebra generated by $\{W_{s\wedge\tau}\,; s \leq t\}$ (τ is a stopping time)

$s \vee t$ - the maximum of s and t (=max(s,t))

$s \wedge t$ - the minimum of s and t (=min(s,t))

E^y - the expectation with respect to measure Q^y

E - the expectation w.r.t a measure which is clear from the context

∂G - the boundary of the set G

$\overline{G}$ - the closure of the set G

G^0 - the interior of the set G

QVI - the quasi-variational inequalities

$\Phi(x)$ - the probability distribution function of the standard normal variable

$\phi(x)$ - the density function of the standard normal variable

CRRA - Constant Relative Risk Aversion type of functions:

$$U(c) = \begin{cases} \frac{c^\alpha}{\alpha} & 0 < \alpha < 1 \\ \log(c) & \alpha = 0 \end{cases}$$

1 Introduction

1.1 Background

Consider a financial market with two assets: a risk free asset, and a stock that behaves like an Ito process. At any instance the investor may rebalance his portfolio by moving capital from stocks to the risk free asset and vice versa in order to maximize an expected utility at some terminal time T.

For common utility functions (power, log), when there are no transaction costs the problem was solved by Merton [19], [20] and the optimal strategy dictates that one has always to keep the same *fixed proportion* of the total wealth invested in stocks. It means that the investor has to rebalance his portfolio all the time and thus the trading strategy is continuous in time.

For the case when proportional transaction costs are present, the problem was also solved with the help of singular stochastic control methods [27], [8] and the policy also demands continuous trading.

But everything changes when fixed transaction costs enter the scene. Paying the same fixed amount of money for every intervention, investor clearly goes to bankruptcy if he chooses the continuous strategy. Now he has to trade at a carefully chosen discrete sequence of instances.

1.2 Goals and Methods

In this work we are going to find the optimal investment strategy of maximizing the expected power utility at some terminal time T when only fixed transaction costs are present. The main method we are using here is called *the impulse control* and was first presented by Benssounssan and Lions, [2] and also by Eastham and Hastings [9] . It helps us to build the system of inequalities and by solving them one can obtain the optimal strategy. The main difficulty is that these inequalities are very hard to solve. So we are trying to approximate the solution and thus to find approximated optimal strategy. The approximation methods we are using were first presented by Atkinson and Wilmott [1], Whalley and Wilmott [29], [30] and also by Korn [13], [14] to solve impulse control problems.

1.3 Organization of the work

The rest of the work is organized as follows: in Section 2 we provide the literature survey on continuous-time stochastic model presented by Merton and his way of solving the utility maximization problem in the no-transactions case. Then we'll go to the case when proportional transactions are present and describe the optimal policy obtained by Norman and Davis. In the end of the section we'll formulate the basics of impulse control, mainly based on the Oksendal's presentation as well as the survey on the recent works on portfolio optimization with fixed transaction costs. In Section 3 we'll formulate the problem exactly for our case and then present an idea about what can be done in order to apply an approximation procedure. After solving the problem we discuss some features of the optimal policy. In Section 4 we'll discuss the Goal Problem, it's solution for the no-transactions case, where we'd like to present a possible different derivation. Also we apply the approximation method from the previous section in order to solve the Goal Problem for the fixed transactions case. In Section 5 we'll provide few simulations in order to demonstrate the performance of the optimal policy. In Section 6 we'll discuss the results and the possible directions for future research.

2 Literature survey

We provide an overview of papers and methods regarding continuous stochastic models in finance and portfolio optimization problems with or without transaction costs:

i. The first Merton model (no transaction costs).

ii. The models with proportional transaction costs.

iii. Impulse control .

iv. The models with fixed transaction costs.

2.1 The original Merton model (no transaction costs)

In his seminal papers, Merton [19], [20] introduced an optimal portfolio management model of a single agent in a stochastic setting. Trading takes place between a riskless security (e.g. a bond) and one or more stocks whose prices are modelled as *diffusion processes*. For each stock, the mean rate of return and volatility are assumed to be constant and known. The investor, endowed with some initial wealth, trades dynamically between the available securities and may also consume part of his wealth continuously in time. The investor is assumed to be a "small" one, in the sense that his actions do not influence the equilibrium prices of the underlying assets. His objective is to maximize the expected utility function which models his individual preferences as well as his attitude towards the risk associated with the market uncertainty. In Merton's work the so called Constant Relative Risk Aversion (CRRA) utility functions were used:

$$\begin{cases} U(c) = \frac{c^\alpha}{\alpha} & 0 < \alpha < 1 \\ U(c) = \log(c) & \alpha = 0 \end{cases}$$

In the rest of the work we take $0 < \alpha < 1$ if not defined otherwise and we'll refer to such utility as *power* or *Merton* utility.

Current prices, but no other information, are available to the investor, there are no transaction costs and the assets are infinitely divisible. In this idealized setting, Merton was able to derive a closed form solution to the stochastic control problem faced by the investor.

Consider a market with two securities: bond without interest rate and a stock, whose prices $b(t)$ and $p(t)$ satisfy

$$\begin{cases} db(t) &= 0 \\ dp(t) &= p(t)(\mu dt + \sigma dW_t) \end{cases}$$

with some $b(0) > 0$ and $p(0) > 0$, where

- $\mu > 0$ - drift-mean rate of return
- $\sigma > 0$ - volatility
- W_t - standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$.

The wealth process satisfies $X(t) = B(t) + S(t)$ with the amounts $B(t)$ and $S(t)$ representing the current holdings in the bond and the stock accounts. The state wealth equation is given simply as

$$dX(t) = S(t)(\mu dt + \sigma dW_t) \tag{2.1}$$

The wealth process must satisfy the state constraint

$$X(s) \geq 0 \ \text{ a.s. } \ \text{for } t \leq s \leq T$$

for some fixed terminal time T and initial t.

The control $S(t) = S(t, X(t))$ is admissible if it is $\mathcal{F}_t$ - progressively measurable - with $\mathcal{F}_s = \sigma(W_u \ ; t \leq u \leq s)$, it satisfies $E\Big[\int_t^T S(s)^2 ds\Big] < +\infty$ and state constraint from above is satisfied. Let us denote by $\mathcal{A}$ the set of all admissible policies.

The aim is to maximize an expected power utility at some terminal time T, i.e. to find

$$w(x,t) = \sup_{\mathcal{A}} E\Big[U(X(T)) \ \mid X(t) = x\Big] = \sup_{\mathcal{A}} E\Big[\frac{(X(T))^\alpha}{\alpha} \ \Big| \ X(t) = x\Big]$$

Using stochastic analysis and under appropriate regularity and growth conditions on the value function, we get that w solves the associated HJB equation

$$\begin{cases} w_t + \sup_S \left[\frac{1}{2}\sigma^2 S(t)^2 w_{xx} + \mu S(t) w_x\right] = 0, \\ w(0,t) = 0 \ \text{ for } t \in [0,T), \\ w(x,T) = \frac{x^\alpha}{\alpha}. \end{cases}$$

Such that the solution is given as:

$$w(x,t) = \frac{x^\alpha}{\alpha} e^{\lambda(T-t)}, \quad \text{where} \quad \lambda = \frac{\mu^2 \alpha}{2\sigma^2(1-\alpha)}.$$

The optimal investment strategy, which is given as

$$S^*(t, X(t)) = \frac{\mu}{\sigma^2(1-\alpha)} X(t)$$

dictates to keep the same *fixed proportion* of the total wealth invested in stocks. It means that it's stationary and it says to keep the point (X, S) on the line, known as "Merton line". It means that the investor has to rebalance his portfolio all the time and thus the trading strategy is continuous in time.

One important generalization of the basic Merton model which may serve as introduction to the next subsection is the *intermediate consumption* case.

Assume that trading takes place in an infinite horizon. The investor consumes at rate $C(t)$ from the bank account. The state wealth equation now is given as

$$dX(t) = S(t)(\mu dt + \sigma dW_t) - C(t)dt.$$

Utility comes only from this consumption and the value function is defined as the maximal expected discounted utility:

$$V(x) = \sup_A E\Big(\int_0^{+\infty} e^{-\rho t} U(C(t)) dt | X(0) = x \Big).$$

The set of admissible policies A is now a pair $(S(t), C(t)) = (S(t, X(t)), C(t, X(t)))$, which are $\mathcal{F}_t$ - measurable and satisfy the integrability conditions:

$$E \int_0^T S^2(s) ds < \infty \,, \quad E \int_0^T C(s) ds < \infty \,, \quad \forall T > 0 \,, \quad \text{as well as } X(s) \geq 0$$

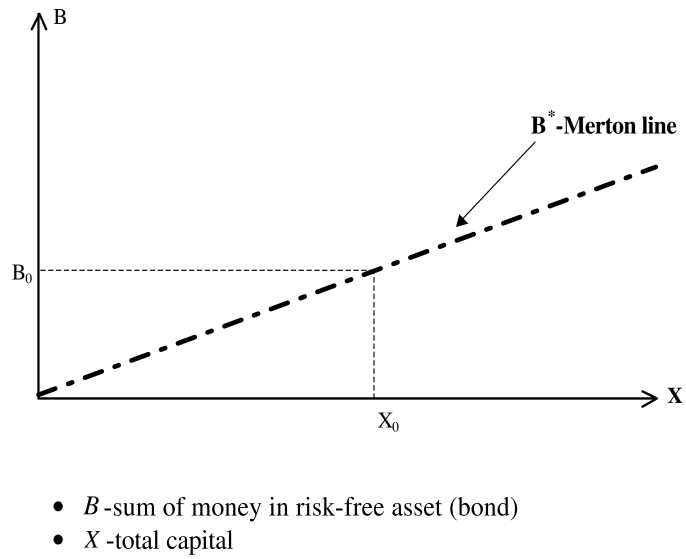
The optimal control strategy is given as

$$S^*(t, X(t)) = S^*(X(t)) = \frac{\mu}{\sigma^2(1-\alpha)} X(t) \text{ and } C^*(t, X(t)) = \Upsilon X(t)$$

for some positive uniquely determined constant Υ.

The consumption argument is out of interest in our work, what's important is that the investor has to rebalance his portfolio position all the time to match the *Merton line*.

Figure 2.1: Merton optimal strategy in the *no transaction costs* case



There are many more generalizations of the Merton models in the absence of transaction costs case, for instance models with non-linear stock dynamics. The interested reader may refer to the brilliant book [12] for further details.

2.2 The models with proportional transaction costs

A crucial simplification in Merton's work is the absence of *transaction costs*. In such idealized model the investor would optimally maintain a proportion of wealth in the stock by trading *continuously*. Such continuous strategies are no longer admissible once proportional transaction costs are introduced. The investor must then determine when the stock position is sufficiently "out of line" to make the trading worthwhile. The first to incorporate the proportional transaction costs in Merton's model were Magil and Constantinides [18], [7]. They brought out an important insight about the policies of different nature, the so called *singular trading policies*. Such policies are characterized by instantaneous trading at the boundaries of a "no-transaction" region whenever the stock position falls on these boundaries. Although Magil and Constantinides did not provide a rigorous *singular stochastic control* formulation of the underlying model, they paved the way to the correct formulation of the valuation models with transaction costs.

Taksar, Klass and Assaf [28] were the first to formulate a transaction cost model as a singular stochastic control problem in the context of maximizing the long term expected rate of wealth growth, i.e. the problem consists of maximizing

$$E\{\liminf_{t\to\infty} \frac{1}{t}[\ln X(t)]\}$$

with the total wealth $X(t) = B(t) + S(t)$ as defined before.

Davis and Norman [8] were the first to provide a rigorous mathematical formulation and extensive analysis of the Merton problem in the presence of proportional transaction costs for CRRA utilities. Their paper is considered a landmark in the literature of transaction costs and contains useful insights and fundamental results, both theoretical and numerical, for the value function and the optimal investment policy. The survey we provide here is based on Chapter 12 of [12].

The investor holds $B(t)$ dollars of the bond and $S(t)$ dollars of the stock at time t. Consider a pair of right-continuous with left limits, non-decreasing processes $(L(t), M(t))$ such that $L(t)$ represents the cumulative dollar amount transferred into the stock account and $M(t)$ the cumulative dollar amount transferred out of the stock account. By convention, $M(0) = L(0) = 0$. The stock account process is

$$S(t) = S + \int_0^t \mu S(s)ds + \int_0^t \sigma S(s)dW_s + L(t) - M(t),$$

with $S(0) = S$. Transfers between the stock and the bond accounts incur *proportional transaction costs*. In particular, the cumulative transfer $L(t)$ into the stocks reduces the bond account by $\beta L(t)$ and the cumulative transfer $M(t)$ out of the stock account increases the bond account by $\lambda M(t)$, where $0 < \lambda < 1 < \beta$. The investor consumes at the rate $C(t)$ dollars out of the bond account. There are no transaction costs in transfers from the bond account into the consumption good. The bond account process is (assuming also the interest rate r for bond)

$$B(t) = B + \int_0^t (rB(s) - C(s))d\tau - \beta L(t) + \lambda M(t),$$

with $B(0) = B$. The integral represents the accumulation of interest and the drain due to consumption.

The last two terms represent the cumulative transfers between the stock and bond accounts.

Define the state space D for a process $(B(t), S(t))$ as

$$D = \{(B,S) \in R \times R : B + \binom{\lambda}{\beta} S \geq 0$$

with the notation

$$\binom{\lambda}{\beta} z = \begin{cases} \lambda z & \text{if } z \geq 0 \\ \beta z & \text{if } z \leq 0 \end{cases}$$

To put it simple, just say that the policy now is a $\mathcal{F}_t$-progressively measurable triple $(C(t), L(t), M(t))$. And the value function V is defined to be

$$V(B,S) = \sup_{\mathcal{A}} E\Big(\int_0^{+\infty} e^{-\rho t} U(C(t))dt | B(0) = B,\ S(0) = S\Big).$$

The optimal trading strategy is as follows: there are two unique numbers z_1 and z_2. When the ratio of account holdings $\frac{S}{B}$ is between the threshold levels z_1 and z_2, then it is optimal not to rebalance the portfolio but only to consume. In other words, the individual must refrain from trading in the region $NT = \{(B,S) \in D : z_1 \leq \frac{S}{B} \leq z_2\}$.

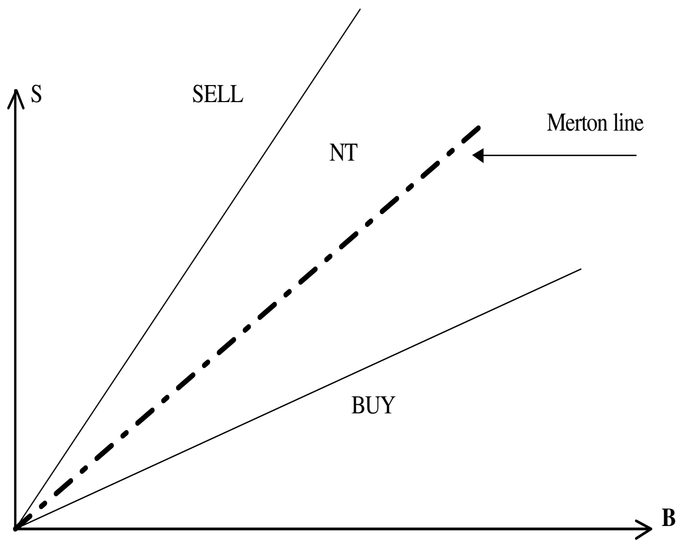
If the holdings ratio $\frac{S}{B}$, is below z_1 then it's optimal to instantaneously rebalance the portfolio components by moving from the original point to the point $(\bar{B}, \bar{S})$ with $\bar{S} = z_1 \bar{B}$ and $\bar{B} = \frac{B+\beta S}{1+\beta z_1}$. This corresponds to a transaction of *buying shares of stock* and this is the optimal policy that one should apply to all points $(B,S) \in \overline{D}$ with $\frac{S}{B} < z_1$. Similarly, if the holdings ratio $\frac{B}{S}$ is above z_2, then it is optimal to instantaneously rebalance the portfolio components by moving to the point $\tilde{S} = z_2 \tilde{B}$ with $\tilde{B} = \frac{B+\lambda S}{1+\lambda z_2}$. This corresponds to a transaction of *selling stock shares* and this is the optimal policy for all points $(B,S) \in \overline{D}$ such that $\frac{S}{B} > z_2$.

So the state space $\overline{D}$ depletes onto three regions: the so-called *SELL* , *BUY* regions (sales and purchases of stock shares occur instantaneously) and the *no trading (NT)* region. The *NT* region lies between the *BUY* and the *SELL* region and the common boundaries are straight lines emanating from the origin.

Thus it is optimal to make transactions corresponding to local time at $\partial(NT)$, resulting in reflection back to NT every time $(B(t), S(t)) \in \partial(NT)$.

It can be seen on the (B, S) plane in the picture.

Figure 2.2: Optimal strategy with *proportional* transaction costs



- *B* -sum of money in risk-free asset (bond)
- *S* - sum of money in stock

Another famous work on this subject was Shreve and Soner [27], who studied the same model and extended the results of Davis and Norman [8] in several directions.

2.3 The basics of Impulse Control

When, in addition to proportional transaction costs, the investor is facing fixed ones, the optimal policy changes dramatically. Now he has to choose a sequence of intervention times and to trade only at these times and not at every instant as it was before, because the fixed component in transaction fee can lead such a policies to bankruptcy.

The technique that helps us to deal with such kind of stochastic control problems is called *impulse control* and was first presented by [2]. Later on it was applied to the basic models in finance in [9]. Here we'll follow the more updated version of Oksendal's lecture notes [22].

Note that the letter S in this subsection is used only to indicate the state space of the process and not money invested in stock.

Suppose that, if there are no interventions, the state $Y(t) \in R^k$ of the system satisfies stochastic differential equation of the form

$$dY(t) = \mu\Big(Y(t)\Big)dt + \sigma\Big(Y(t)\Big)dW_t \,; Y(0) = y \in R^k,$$

where $\mu : R^k \longrightarrow R^k$ and $\sigma : R^k \longrightarrow R^{k\times d}$ are given continuous functions and W_t is a d-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with $W_0 = 0$ a.s. The differential operator L which coincides with the generator of Y is given by

$$L\psi(y) = \sum_{i=1}^{k} \mu_i(y)\frac{\partial \psi}{\partial y_i} + \frac{1}{2}\sum_{i,j=1}^{k}(\sigma\sigma^{tr})_{ij}(y)\frac{\partial^2 \psi}{\partial y_i \partial y_j},$$

defined for all functions $\psi : R^k \longrightarrow R$, which are twice differentiable at $y \in R^k$. Suppose that at any time t and at any state y we are free to intervene and give the system an *impulse* $\zeta \in Z$, where $Z = Z(y)$ is a given set which may depend on y (the set of admissible impulse values). Suppose that the result of giving such an impulse when the state of the system is y is that the state jumps immediately from $Y(t^-) = y$ to $Y(t) = \Gamma(y, \zeta)$, where $\Gamma : R^k \times Z \longrightarrow R^k$ is a given function.

An *impulse control* for this system is a double (possibly finite sequence)

$$v = (\tau_1, \tau_2, ..., \tau_j, ...; \zeta_1, \zeta_2, .., \zeta_j, ..)_{j\leq N}; \quad N \leq \infty$$

where $\tau_1 < \tau_2 < ...$ are $\mathcal{F}_t$ - stopping times and $\zeta_j, j \geq$ are $\mathcal{F}_{\tau_j}$ - measurable random variables representing the corresponding impulses, $\zeta_j \in Z$. We interpret $\tau_1,\ \tau_2, ...$ as the *intervention times*, i.e. the times when we decide to intervene and give the system the impulses $\zeta_1,\ \zeta_2, ..$ respectively. $N \leq \infty$ is the number of interventions.

If $v = (\tau_1, \tau_2, ..., \tau_j, ...; \zeta_1, \zeta_2, .., \zeta_j, ..)$ is applied to the system $Y(t)$, it gets the value $Y^{(v)}(t)$ which inductively can be described as follows :

$$dY^{(v)}(t) = \mu\Big(Y^{(v)}(t)\Big)dt + \sigma\Big(Y^{(v)}(t)\Big)dW_t \ , \quad \tau_j < t < \tau_{j+1} \leq \widetilde{T}$$

$$Y^{(v)}(\tau_{j+1}) = \Gamma\Big(Y^{(v)}(\tau_{j+1}^-), \zeta_{j+1}\Big) \ , \ j = 0, 1, 2, ...; \ \tau_{j+1} \leq \widetilde{T},$$

where we put $\tau_0 = 0$ and $\widetilde{T}$ is the first time our system exits its state space : Let $S \subset R^k$ be a fixed Borel set such that $S \subset \overline{S^0}$, where S^0 is the interior of S, $\overline{S^0}$ its closure. Define

$$\widetilde{T} = \widetilde{T}_y^{(v)}(\omega) = \inf\{t \geq 0 \ : \ Y^{(v)}(t, \omega) \notin S\}.$$

Note that this is the general definition and in our - fixed interval - case, this $\widetilde{T}$ will be either the T- the end of the interval, or the first time our process X becomes zero.

Let $Q^y = Q^{y,v}$ denote the law of the stochastic process $Y^{(v)}(t)$ starting at $Y^{(v)}(0) = y$.

We now describe the performance criterion for our system:

Let $g : \partial S \rightarrow R$ be a given *utility* function , where ∂S is the boundary of S. We assume that we are given a set V of *admissible controls* which includes the set of impulse controls $v = (\tau_1, \tau_2, ..., \tau_j, ...; \zeta_1, \zeta_2, .., \zeta_j, ..)$, such that a unique strong solution $Y(t) = Y^{(v)}(t)$ exists and

$$Y^{(v)}(t) \in S \text{ for all } t < \widetilde{T} \ \text{ and}$$

$$\lim_{j \longrightarrow \infty} \tau_j = \widetilde{T} \ \ Q^y \text{ a.s.} \quad \text{ for all } y \in R^k$$

(if $N < \infty$ we assume that $\tau_N = \widetilde{T}$ a.s.).

From now on we assume that

$$E^y\Big[|g(Y^{(v)}(\widetilde{T})|1_{\widetilde{T}<\infty}\Big] < \infty \quad \text{for all } y \in R^k, v \in V.$$

Where E^y represents the expectation with respect to Q^y.

If $v \in V$ then the *performance* or *utility* is defined by

$$w^{(v)} = E^y\Big[g(Y^{(v)}(\widetilde{T})1_{\widetilde{T}<\infty}\Big].$$

So we are now to formulate the impulse control problem:

Find the value function $\Phi(y)$ and an optimal admissible impulse control $v^* \in V$, such that

$$\Phi(y) = \sup\{w^{(v)}\,; v \in V\} = w^{(v*)}.$$

Let H denote the space of all measurable functions $h : S \to R$.

The *intervention operator* M: $H \to H$ for all $h \in H$ and $y \in S$ is defined by

$$Mh(y) = \sup\{h(\Gamma(y,\zeta))\,; \zeta \in Z \text{ and } \Gamma(y,\zeta) \in S\}.$$

Suppose that for each $y \in S$ there exists at least one $\hat{\zeta} = \hat{\zeta}(y) \in Z$ such that the supreme above is obtained, i.e. $\hat{\zeta}(y) \in$ Argmax$\{h(\Gamma(y,\cdot))\}$, and that a measurable selection $\hat{\zeta} = K_h(y)$ of such maximum points exists. Then we have

$$Mh(y) = h\Big(\Gamma(y, K_h(y))\Big)\ ,\ y \in S.$$

Quasi-variational verification theorem for impulse control

a) Suppose that we can find a function $\varphi : \bar{S} \to R$ such that

1. $\varphi \in C^1(S^0) \cap C(\bar{S})$;
2. $\varphi \geq M\varphi$ on S;
3. Define

   $$D = \{y \in S\,; \varphi(y) > M\varphi(y)\} - \text{the continuation region}$$

   and suppose that $Y(t)$ spends 0 time on ∂D a.s., i.e.

   $E^y\Big[\int_0^{\widetilde{T}} 1_{\partial D}(Y(t))dt\Big] = 0$ for all $y \in S$;
4. Suppose that ∂D is a Lipschitz surface, i.e. is locally the graph of a function h, such that there exists $K < \infty$ with

   $$|h(x) - h(y)| \leq K|x - y| \quad \text{for all } x, y;$$
5. $\varphi \in C^2(S^0 \backslash \partial D)$ and the second order derivatives of φ are locally bounded near ∂D;

6. $L\varphi \leq 0$ on $S^0 \setminus \partial D$;

7. $\varphi(Y(t)) \longrightarrow g(Y(\widetilde{T}))1_{\{\widetilde{T}<\infty\}}$ as $t \to \widetilde{T}$ a.s. Q^y for all $v \in V$, $y \in S$;

8. The family $\{\varphi^-(Y(\tau))\,; \tau -$ stopping time, $\tau \leq \widetilde{T}\}$ is uniformly integrable w.r.t. Q^y for all $v \in V$, $y \in S$. Recall that $\varphi^-(y) = \max(0, -\varphi(y))$.

Then

$$\varphi(y) \geq \Phi(y) \text{ for all } y \in S.$$

b) Suppose that in addition to 1.- 8. above,

1. $L\varphi = 0$ on $D \cap S^0$;

2. $\hat{\zeta}(y) = K_\varphi(y)$ exists for all $y \in S$.

   Define the impulse control

$$\hat{v} := (\hat{\tau}_1, \hat{\tau}_2, ...; \hat{\zeta}_1, \hat{\zeta}_2, ...)$$

   as follows:
   Put $\hat{\tau}_0 = 0$ and define inductively

$$\hat{\tau}_{j+1} = \inf\{t > \hat{\tau}_j\,; Y^{(j)}(t) \notin D\} \wedge \widetilde{T}$$

   and

$$\hat{\zeta}_{j+1} = K_\varphi(Y^{(j)}(\hat{\tau}_{j+1}^-))\ , \text{ if } \hat{\tau}_{j+1}^- < \widetilde{T}\ ,$$

   where $Y^{(j)}(t)$ is the result of applying the impulse control
   $\hat{v}_j := (\hat{\tau}_1, \hat{\tau}_2, ..., \hat{\tau}_j; \hat{\zeta}_1, \hat{\zeta}_2, ..., \hat{\zeta}_j)$ to Y.

3. Suppose that $\hat{v} \in V$ and the family $\{\varphi(Y^{(\hat{v})}(\tau))\,; \tau$ - stopping time, $\tau \leq \widetilde{T}\}$ is uniformly integrable w.r.t. Q^y for all $v \in V$, $y \in S$.

Then

$$\varphi(y) = \Phi(y) \text{ for all } y \in S$$

and $\hat{v}$ is optimal.

2.4 The models with fixed transaction costs.

Clearly one has to modify the verification theorem from above in order to suit the models in finance. For our case we will do all the work in the next section. Here we describe some works which apply impulse control in finance. The first who applied the impulse control method were Hastings and Eastham [9]. They have formulated the problem in more rigorous setting for the case of finite horizon (which is our case as well) and proposed a method of solution via iterated optimal stopping. Their aim was to maximize the expected wealth at terminal time T, i.e. their utility function was $U(x) = x$.

Oksendal and Sulem, [23], [24] have made a big contribution to this theory and there are many interesting examples in their lecture notes [22] and papers [23], [24], where they have tried to generalize the results of Davis and Norman [8], Shreve and Soner [27] for the case when there is also a fixed component in the transaction cost.

One of the recent papers on the subject is a very interesting work by Pliska and Suzuki [26], where they have studied the asset allocation problem of *optimal tracking* a target mix of asset categories for the case of fixed and proportional transaction costs. Formulating the problem with infinite horizon, they have succeeded in solving the QVI inequalities explicitly. They have shown that when the value function depends only on the fraction process $Y_t = \frac{S_t}{B_t + S_t}$ it is possible to reduce the problem to some solvable equation in one dimension. The optimal investment strategy that they have got is of particular interest and will be discussed in the next section as well.

The common problem with the impulse control method is that it is extremely difficult to find an explicit solution of the QVI, especially when the horizon is finite. So there were attempts to find some approximation method of solution. The method that we use in this work was introduced by Atkinson and Willmott [1], Whalley and Willmott [29], [30] and further applied by Korn [13], [14] to solve such portfolio optimization problems.

The problem studied in Willmott's papers is significantly different from ours. It deals with optimal hedging rather than optimal portfolio selection, in the presence of fixed transaction costs. The most remarkable feature was introducing this concept of expanding the value function in the power series of ε.

For the sake of this two main things were introduced: first, the transaction cost was assumed to be a function of ε, which goes to zero as ε goes to zero; and second, knowing the optimal strategy in the case of no transaction costs, a single additional variable can be introduce to actually measure the difference between the current state of the system and the "optimal" one.

The utility function they use is the negative exponential function

$$U(x) = 1 - exp(-\gamma x).$$

The control taken in their case was the number of shares (stocks) held in the portfolio, defined by y, and it was known that in the no - transaction costs case the optimal thing was *always* to keep the same *fixed* number of shares, say y^* in the portfolio. So the rescaling variable Y was introduced by :

$$y = y^* + \varepsilon^{1/4} Y.$$

The control proposed is actually to trade in the selected time instances and not continuous trading, since transaction cost have a fixed component. Then they assume optimal restarting lines $y^* + \varepsilon^{1/4} Y^+$ and $y^* + \varepsilon^{1/4} Y^-$ as well as optimal boundaries $y^* + \varepsilon^{1/4} \hat{Y}^+$ and $y^* + \varepsilon^{1/4} \hat{Y}^-$. These boundaries uniquely determine the trading strategy - one has to trade in the very moment that the process reaches the boundary and the action to be taken is to move the process to the correspondent restarting lines.

Then the equation that corresponds to optimal hedging was introduced. The value function was viewed as an expansion in the power series of ε and all the boundaries were found. However, their hedging problem is unrelated to ours and hence we will not elaborate further.

We are interested in the optimization the utility of the terminal wealth and it was Korn [13], [14] who combined the method described above with the simplified impulse control statement developed in Eastham and Hastings [9], to solve this kind of problem. The utility function that was taken was the negative exponential, as in Willmott's work.

All rescalings in Korn were just the same as in Willmott, because the optimal policy in the no - transaction costs case was again to keep the same fixed number of shares in the portfolio.

The function $e^{-\gamma(S_T+B_T)}$, looks rather friendly, because the bond and stock can be separated in the sense that

$$e^{-\gamma(B+S)} = e^{-\gamma B} e^{-\gamma S},$$

and this leads to some significant simplifications. For example, omit the "1" term from the negative exponential and write the value function as

$$v(t, B, S) = \max_V E\Big(- e^{-\gamma(B_T+S_T)} | S_t = S, B_t = B\Big).$$

It possible to show that

$$v(t, B, S) = e^{-\gamma B} v(t, 0, S),$$

and so the optimal strategy is expected to be *independent of the total wealth* - a feature that is clearly unrealistic, but simplifies the way in which fixed transaction payments should be treated.

Hence he ignores the total wealth and separates the stock price p and the number of shares y (which remains fixed between interventions) by introducing function

$$q(t, p, y) = v(t, 0, py).$$

Then QVI inequalities were stated for the function q and only after that the rescaling variable Y has entered the scene and one has to work hard in order to keep track of all partial derivatives that appear on the way. As Whalley and Wilmott, Korn has also succeeded in finding the optimal boundaries, which were given as

$$(Y^+, \hat{Y}^+) = \Big(0, \sqrt[4]{\frac{12 r \mu^2}{\gamma^3 \sigma^4 p^4}}\Big) \text{ and } (Y^-, \hat{Y}^-) = -(Y^+, \hat{Y}^+)$$

where the fixed transaction cost was equal to $r\varepsilon$.

It is worth noting that the problem was actually solved in a rather complicated manner and without any clue on how one can generalize the ideas in order to apply it to more realistic utility functions and other rescaling procedure.

In the present work we generalize the method described above with the help of a more general impulse control theory, to be found for instance in the recently written lecture notes of Oksendal [22].

We argue that by knowing the solution of the problem in the no - transaction costs case, we can approximate the solution for the case of transaction costs (fixed, possibly with an addition of proportional ones). By rescaling the original process we introduce a new diffusion process and apply the impulse control theory to this new process. This is done in a very explicit and friendly way, in contrast to the previous works on the subject.

In order to demonstrate the method, first we solve for power and logarithmic utility functions. Then we solve the Goal Problem - and the method proves to be rather effective.

3 Maximizing Merton utility

This is the main part of the work. Here we'll use the impulse control and approximation methods to find the optimal strategy of optimizing the power utility function for the case of fixed transaction costs.

3.1 Problem formulation

First of all we'd like to apply the impulse control technique to our case, so let's concretize the problem for the finite horizon.

The stochastic process now is 3-dimensional $Z(t) = (t, S(t), B(t))$ defined in the region:

$$t \in (0, T], \quad (S(t), B(t)) \in \{(S, B) \in R \times R, \; : S + B \geq 0\}.$$

The differential operator L, which coincides with the generator of the process $(t, S(t), B(t))$ is given for all sufficiently smooth functions $\phi(t, S, B)$ as

$$L\phi(t, B, S) := \frac{1}{2}\sigma^2 S^2 \phi_{SS}(t, S, B) + \mu S \phi_S(t, S, B) + \phi_t(t, S, B).$$

Define by $\tau_1, \tau_2, ...$ - the intervention times and $\zeta_1, \; \zeta_2, ..$ - sums of money transferred from bond to stock at correspondent interventions.

Denote by k the transaction fee, i.e. some fixed amount of money that investor has to pay each time he chooses to intervene.

Denote by V the set of all admissible impulse control strategies. If $v \in V$ is applied to the process $Z(t)$, it behaves according to

$$\begin{cases} dS(t) = S(t)(\mu dt + \sigma dW_t), \quad \tau_{j-1} \leq t < \tau_j \leq T; \\ dB(t) = 0, \quad \tau_{j-1} \leq t < \tau_j \leq T; \\ B(\tau_j) = B(\tau_j^-) - k - \zeta_j; \\ S(\tau_j) = S(\tau_j^-) + \zeta_j. \end{cases} \tag{3.1}$$

and the expected terminal utility under the policy v is given as

$$w^{(v)}(t, S, B) = E\Big[U\Big(S(T) + B(T)\Big)\Big|S(t) = S, B(t) = B\Big],$$

with the same CRRA utility function :

$$\begin{cases} U(c) = \frac{c^\alpha}{\alpha}, & 0 < \alpha < 1; \\ U(c) = \log(c), & \alpha = 0. \end{cases}$$

where we'll look at $0 < \alpha < 1$ or *power (Merton) utility* case.

The main objective is to find the optimal trading strategy v^* and the maximal expected utility

$$W(t, S, B) = w^{(v^*)}(t, S, B) = \sup_{v \in V} w^{(v)}(t, S, B).$$

For every measurable function $w(t, S, B)$ the *maximum or intervention operator* is defined as

$$Mw(t, S, B) = \max_{\zeta} \{w(t, S + \zeta, B - \zeta - K)\}.$$

If the optimal impulse control v^* is already chosen, then clearly holds:

$$w^{(v^*)}(t, S, B) \geq Mw^{(v^*)}(t, S, B),$$

because it's not always optimal to trade at time t. But when it's optimal, both parts in above inequality have to coincide.

To show uniform integrability, note that the Merton solution in the no-transaction costs case is always better than our function

$$E^{t,S,B}\Big[w(\tau, S_\tau, B_\tau)\Big] \leq E^{t,S,B}\{\frac{(B_T + S_T)^\alpha}{\alpha}\} < \infty,$$

and the uniform integrability follows.

The *quasi-variational inequalities* for our case can be stated as

$$\begin{cases} Lw(t, S, B) \leq 0; \\ w(t, S, B) \geq Mw(t, S, B); \\ \Big(w(t, S, B) - Mw(t, S, B)\Big) Lw(t, S, B) = 0; \\ w(T, S, B) = U(S + B); \\ w(t, S, B) 1_{\{B+S=0\}} = 0. \end{cases} \tag{3.2}$$

The last condition can be changed into $w(t, S, B)1_{\{B+S \leq k\}} = 0$ and it is just a matter of choice - how to define the behavior of the process in the case when transaction cost can exceed the total wealth.

Clearly the solution of this system defines the *optimal strategy* in the following way: define the *continuation region*

$$NT = (t, S, B) : \{w(t, S, B) > Mw(t, S, B)\}$$

and assume that we start in NT. In the very moment that our process $\big(t, S(t), B(t)\big)$ exits NT we have to intervene in an *optimal way*, in order to move the system back to NT.

Here we'd like to say that this quasi-variational inequalities (QVI) theorem is the verification statement. It means that if we can find a function with the following properties then this function is the optimal value function and the control defined is the optimal. The problem is that we don't know the function w, but, like in most of the optimal stopping problems, by assuming the form of the strategy we can make a big step forward in solving the inequalities.

3.2 Approximation scheme

The remarkable property of power utility is that without transaction costs the optimal strategy is to keep the same *fixed proportion* of the total wealth invested in the risk free asset, i.e. always to keep the sum $B^*(t) = (1-u)X(t)$ in bond (Merton strategy), with $u = \frac{\mu}{\sigma^2(1-\alpha)}$.

The idea is to assume that fixed transaction cost is a function of a small parameter ε, which goes to zero as ε goes to zero, and then to find the optimal trading strategy which depends on ε and which converges to Merton strategy when ε goes to zero.

The transaction cost is assumed to be

$$k = r\varepsilon^4, \text{ for some const } r > 0.$$

(see Appendix for more explanation).

In the no transaction region the value function will be viewed as an expansion in powers of ϵ. The natural point around which such an expansion should be made is the amount of money invested in bond in the Merton strategy:

$$B^*(t) = (1-u)X(t).$$

Rescale our variables by introducing the new variable $Y = Y(t)$:

$$\begin{cases} B(t) = B^*(t) + \varepsilon Y(t) = (1-u)X(t) + \varepsilon Y(t); \\ S(t) = X(t) - B(t) = uX(t) - \varepsilon Y(t). \end{cases}$$

Y actually measures or indicates how far is the portfolio position from the Merton line. And we are going to assume that there are upper and lower boundaries $\widehat{Y}^+$ and $\widehat{Y}^-$, such that when Y is between them - no transactions should be made, but when Y exits these boundaries, then we have to intervene and move it back to some optimal restarting values Y^+ and Y^-. Clearly such strategy converges to Merton strategy when ε goes to zero.

We now formulate our assumptions more rigorously:
assume that there exist some functions $\widehat{Y}^+$, $\widehat{Y}^-$, Y^+, Y^-, which may depend on X and on t, so that

- the no-transaction region (NT) has the form

$$NT = \{(t, x, y) : \widehat{Y}^- < y < \widehat{Y}^+\},$$

- the upper and the lower boundaries of NT are

$$\widehat{B}^+ = B^* + \varepsilon \widehat{Y}^+ \quad \text{and} \quad \widehat{B}^- = B^* + \varepsilon \widehat{Y}^-,$$

- the upper and the lower optimal restarting lines are:

$$B^+ = B^* + \varepsilon Y^+ \quad \text{and} \quad B^- = B^* + \varepsilon Y^-$$

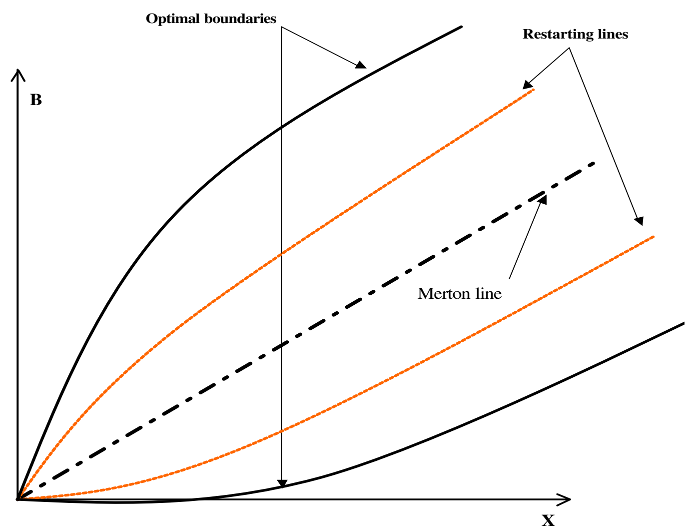
(the signs of $\widehat{Y}^-$and Y^- are negative).

Define the optimal policy as follows:

1. Don't do anything in the NT region,

2. If $(t, X(t), Y(t))$ reaches either upper or lower boundary $(t, X, \widehat{Y}^+)$ or $(t, X, \widehat{Y}^-)$ the investor has to make a transaction and to move it back to the corresponding restarting lines $(t, X(t) - k, Y^+)$ or $(t, X(t) - k, Y^-)$, inside the NT region.

Figure 3.1: Optimal strategy with *fixed* transaction costs



So actually now we have a different process

$$(t, S(t), B(t)) \to (t, X(t), Y(t)),$$

Denote as before $\tau_1, \tau_2, ...$ - intervention times and $\Delta_y^1,\ \Delta_y^2, ..$ - corresponding impulses.

Denote by V the set of all admissible impulse control strategies. If $v \in V$ is applied to the new process, it behaves according to

$$\begin{cases} dX(t) = \Big(uX(t) - \varepsilon Y(t)\Big)\Big(\mu dt + \sigma dW_t\Big)\Big) \ , \ \tau_{j-1} \le t < \tau_j \le T; \\ dY(t) = \frac{u-1}{\varepsilon}\Big(uX(t) - \varepsilon Y(t)\Big)\Big(\mu dt + \sigma dW_t\Big) \ , \ \tau_{j-1} \le t < \tau_j \le T; \\ X(\tau_j) = X(\tau_j^-) - k; \\ Y(\tau_j) = Y(\tau_j^-) - \Delta_y^j. \end{cases}$$

The new generator L is given now as

$$L\varphi = \varphi_t + \mu(ux - \varepsilon y)\Big[\varphi_x + \frac{u-1}{\varepsilon}\varphi_y\Big] +$$

$$\frac{1}{2}\sigma^2(ux - \varepsilon y)^2\Big[\varphi_{xx} + 2\frac{u-1}{\varepsilon}\varphi_{xy} + \frac{(u-1)^2}{\varepsilon^2}\varphi_{yy}\Big],$$

and the value function has changed as well

$$w(t, S, B) \to Q(t, x, y).$$

Note that further we'll write x, y instead of X, Y. The main objective is to find the optimal trading strategy v^* and the maximal expected utility

$$Q(t, x, y) = \sup_{v \in V} E^{(t,x,y)} \Big(U(X_T) \Big| X(t) = x, Y(t) = y \Big).$$

Almost everything in the QVI setting (3.2) remains the same. The main thing that has to be changed is the equation

$$Lw(t, S, B) = 0 \quad \text{changed into} \quad LQ(t, x, y) = 0.$$

So the equation $LQ(t, x, y) = 0$ turns to

$$\frac{1}{2}\sigma^2(ux - \varepsilon y)^2 \Big[Q_{xx} + 2\frac{u-1}{\varepsilon} Q_{xy} + \frac{(u-1)^2}{\varepsilon^2} Q_{yy} \Big]$$
$$+ \mu(ux - \varepsilon y) \Big[Q_x + \frac{u-1}{\varepsilon} Q_y \Big] + Q_t = 0. \tag{3.3}$$

Following the idea of approximation, in the NT we expand Q as:

$$Q(t, x, y) = H_0(t, x) + \varepsilon^2 H_2(t, x) + \varepsilon^4 G(t, x, y) + ..$$

(see Appendix for detailed explanation about the way, such a function can be written).

Assume that this Q satisfies all the conditions of QVI, in particular $LQ = 0$ in the no-transaction region. Then we write down the equation $LQ = 0$ with Q expanded in series and order the terms by the powers of ε.

Considering only $O(1)$ and $O(\varepsilon^2)$ equations and using smoothness conditions $(t < T)$:

$$\begin{cases} Q(t, x, y) = Q(t, x - k, Y^+), \quad \forall\, y \geq \widehat{Y}^+; \\ \\ Q_y(t, x, y) = Q_y(t, x - k, Y^+), \quad \forall\, y \geq \widehat{Y}^+. \end{cases} \tag{3.4}$$

as well as *optimality of transaction* condition

$$\begin{cases} \frac{d}{d(\Delta_y)} Q(t, x - k, \widehat{Y}^+ - \Delta_y) \Big|_{\Delta_y = \widehat{Y}^+ - Y^+} = 0, \\ \\ Q_{yy}(t, x - k, Y^+) < 0. \end{cases} \tag{3.5}$$

we are able to obtain $\widehat{Y}^+$ and Y^+.

Just the same work is done for the lower bound.

Note that we are not going to find the function Q as it requires more complex analysis of the various partial differential equations, but luckily, it's possible to obtain the optimal trading strategy, which is actually the more important thing for investors.

3.3 Solving the QVI

The analysis has been done in order to find the simplest possible form of the function that can solve this optimization problem. It was shown that once you've taken the appropriate kind of Taylor expansion, then the y - dependence can appear the first time only in ε^4 - term.

The interested reader is referred to the Appendix for further details.

We are free to choose some friendly form of the Taylor expansion for the value function, because it is clear that the solution must remain the same.

Power form of the value function

Let's take the power form, which turns out to be the most friendly among the forms we test.

$$Q(t,x,y) = \frac{1}{\alpha}\Big(x + \varepsilon^2 H(t,x) + \varepsilon^4 G(t,x,y) + ...\Big)^{\alpha} e^{\lambda(T-t)}. \tag{3.6}$$

In the Appendix it's also shown that that all the coefficients of all odd powers of epsilon should be zero and hence may be ignored.

Recall that equation (3.2) must hold. Hence we obtain

$$Q_t = -\frac{\lambda}{\alpha}(x+\varepsilon^2 H + \varepsilon^4 G)^{\alpha} e^{\lambda(T-t)} + \Big(x+\varepsilon^2 H+\varepsilon^4 G\Big)^{\alpha-1}\Big(\varepsilon^2 H_t + \varepsilon^4 G_t\Big)e^{\lambda(T-t)},$$

$$Q_x = \Big(x+\varepsilon^2 H+\varepsilon^4 G\Big)^{\alpha-1}\Big(1+\varepsilon^2 H_x+\varepsilon^4 G_x\Big)e^{\lambda(T-t)},$$

$$Q_y = \varepsilon^4 G_y\Big(x+\varepsilon^2 H+\varepsilon^4 G\Big)^{\alpha-1} e^{\lambda(T-t)},$$

$$Q_{xx} = (\alpha-1)\Big(x+\varepsilon^2 H+\varepsilon^4 G\Big)^{\alpha-2}\Big(1+\varepsilon^2 H_x+\varepsilon^4 G_x\Big)^2 e^{\lambda(T-t)} +$$

$$+\Big(x+\varepsilon^2 H+\varepsilon^4 G\Big)^{\alpha-1}\Big(\varepsilon^2 H_{xx}+\varepsilon^4 G_{xx}\Big)e^{\lambda(T-t)},$$

$$Q_{xy} = (\alpha-1)\Big(x+\varepsilon^2 H+\varepsilon^4 G\Big)^{\alpha-2}\varepsilon^4 G_y\Big(1+\varepsilon^2 H_x+\varepsilon^4 G_x\Big)e^{\lambda(T-t)} +$$

$$+\Big(x+\varepsilon^2 H+\varepsilon^4 G\Big)^{\alpha-1}\varepsilon^4 G_{xy} e^{\lambda(T-t)},$$

$$Q_{yy} = \varepsilon^4 G_{yy}\Big(x+\varepsilon^2 H+\varepsilon^4 G\Big)^{\alpha-1} e^{\lambda(T-t)} + (\alpha-1)\Big(x+\varepsilon^2 H+\varepsilon^4 G\Big)^{\alpha-2}\varepsilon^8 (G_y)^2 e^{\lambda(T-t)}.$$

Now we have to put all these derivatives into equation (3.2).

Divide all the terms by $\Big(x + \varepsilon^2 H + \varepsilon^4 G\Big)^{\alpha-2} e^{\lambda(T-t)}$.

Now we can sort the equation in powers of ε and get a number of equations possibly simpler to solve.

Also we are going to use the fact that ε in powers larger than 4 are negligible.

$O(1)$ **term** :

$$\frac{1}{2}\sigma^2(\alpha-1)(u^2x^2) + \mu(ux^2) - \frac{\lambda}{\alpha}x^2 = 0,$$

or

$$\frac{1}{2}\sigma^2(\alpha-1)u^2 + \mu u - \frac{\lambda}{\alpha} = 0.$$

This equation doesn't give us any information because it's always true for optimal

$$u = \frac{\mu}{\sigma^2(1-\alpha)} \qquad \text{and} \qquad \lambda = \frac{\mu^2\alpha}{2\sigma^2(1-\alpha)}.$$

$O(\varepsilon^2)$ **term**:

$$\frac{1}{2}\sigma^2(\alpha-1)\Big(y^2 + 2u^2x^2H_x\Big) + \frac{1}{2}\sigma^2\Big(u^2x^3H_{xx} + u^2(u-1)^2x^3G_{yy}\Big)$$

$$+\mu\Big(ux^2H_x + Hux\Big) - 2\frac{\lambda}{\alpha}xH + xH_t = 0.$$

So from $O(\varepsilon^2)$ we can get a following differential equation for G

$$\frac{1}{2}\sigma^2u^2(u-1)^2x^3G_{yy} + \frac{1}{2}\sigma^2(\alpha-1)y^2 + \widehat{N}(H) = 0$$

or

$$G_{yy} = \frac{1-\alpha}{u^2(u-1)^2x^3}y^2 + N(H). \qquad (*)$$

$(*)$ is solved by:

$$G(t,x,y) = \frac{1}{12}M(t,x)y^4 + \frac{1}{2}N(t,x)y^2 + C(t,x)y + D(t,x) \qquad (3.7)$$

where

$$M = M(t,x) = \frac{1-\alpha}{u^2(u-1)^2}\, x^{-3}$$

The exact value of N is not needed and unknown C, D depend on t and x .

It's difficult to find an explicit form of the value function Q, but, using (3.4) and (3.5), we can find Y^+, $\widehat{Y}^+$, Y^-, $\widehat{Y}^-$.

From conditions (3.4) it follows that $\forall\, y \geq \widehat{Y}^+$

$$x + \varepsilon^2 H(t,x) + \varepsilon^4 G(t,x,y) = x - k + \varepsilon^2 H(t, x-k) + \varepsilon^4 G(t, x-k, Y(x-k)^+)$$

Note that the transaction cost k equals to $\varepsilon^4 r$ and that ε^5 and smaller terms are negligible. By using one-term Taylor expansions of all functions around point x we get

$$\begin{aligned} x + \varepsilon^2 H(t,x) + \varepsilon^4 G(t,x,y) &= x - k + \varepsilon^2 H(t,x) - \varepsilon^6 r H_x(t,x) \\ +\varepsilon^4 G(t,x,Y(x)^+) - \varepsilon^8 r\Big(G_x(t,x,Y(x)^+) &+ \frac{u-1}{\varepsilon} G_y(t,x,Y(x)^+)\Big), \end{aligned} \tag{3.8}$$

and hence the following condition

$$\forall\, y \geq \widehat{Y}^+ \quad G(t,x,Y^+) - G(t,x,y) = r. \tag{3.9}$$

In the same way the lower bound (LB) condition is obtained:

$$\forall\, y \leq \widehat{Y}^- \quad G(t,x,Y^-) - G(t,x,y) = r. \tag{3.10}$$

Thus using together (3.9) and (3.10) we see that the odd-power term in G has to disappear, i.e. $C = 0$. And the equation (3.9) with $y = \hat{Y}^+$ turns to be

$$\frac{1}{12} M\Big((Y^+)^4 - (\widehat{Y}^+)^4\Big) + \frac{1}{2} N\Big((Y^+)^2 - (\widehat{Y}^+)^2\Big) = r. \tag{3.11}$$

Now consider the optimality of transaction condition

$$\frac{d}{d(\Delta_y)}\Big(Q(t, x-k, \widehat{Y}^+ - \Delta_y)\Big)\Big|_{\Delta_y = \widehat{Y}^+ - Y^+} = 0, \tag{3.12}$$

where we note that $\widehat{Y}^+ \equiv \widehat{Y}^+(x)$, and $Y^+ \equiv Y^+(x-k)$ - are functions of x and $x-k$ respectively. From the above follows that

$$G_y(t, x-k, Y^+(x-k)) = 0, \quad \forall x > k \qquad \Rightarrow G_y(t, x, Y^+) = 0.$$

From the requirement of the G_y continuity on the upper border, we also get

$$G_y(t, x, \widehat{Y}^+) = 0.$$

Using two conditions from above we can write that

$$G_y(t, x, Y^+) - G_y(t, x, \widehat{Y}^+) = 0 \tag{3.13}$$

or

$$\frac{1}{3}M\Big((Y^+)^2 + Y^+(\widehat{Y}^+) + (\widehat{Y}^+)^2\Big) + N = 0. \tag{3.14}$$

Expressing N from (3.14) and putting it into (3.11) we get

$$(\widehat{Y}^+)^3 - \widehat{Y}^+\Big((Y^+)^2 + Y^+(\widehat{Y}^+) + (\widehat{Y}^+)^2\Big) = 0,$$

from which follows that $Y^+ = 0$.

Putting this into (3.14) we obtain

$$\widehat{Y}^+ = \sqrt[4]{\frac{12r}{M}} = \sqrt[4]{\frac{12ru^2(u-1)^2}{1-\alpha}} \;\; x^{\frac{3}{4}}.$$

From the analogous results on the lower border

$$\frac{1}{3}M\Big((Y^-)^2 + Y^-(\widehat{Y}^-) + (\widehat{Y}^-)^2\Big) + N = 0$$

and

$$G_y(t, x, \widehat{Y}^-) = 0.$$

we obtain that $\widehat{Y}^- = -\widehat{Y}^+$ and $Y^- = -Y^+$. So we can summarize this subsection with

$$\begin{cases} \widehat{Y}^+ = \sqrt[4]{\frac{12ru^2(u-1)^2}{1-\alpha}} \;\; x^{\frac{3}{4}}, \\ \widehat{Y}^- = -\sqrt[4]{\frac{12ru^2(u-1)^2}{1-\alpha}} \;\; x^{\frac{3}{4}}, \\ Y^+ = 0, \\ Y^- = 0. \end{cases} \tag{3.15}$$

Linear form of the value function

Just to make it sure let us try another form of the value function: Take the following linear form:

$$Q(t,x,y) = \frac{x^\alpha}{\alpha} e^{\lambda(T-t)} + \varepsilon^2 H(t,x) + \varepsilon^4 G(t,x,y) + ..$$

The derivatives are as follows:

$$Q_t = -\lambda \frac{x^\alpha}{\alpha} e^{\lambda(T-t)} + \varepsilon^2 H_t + \varepsilon^4 G_t,$$

$$Q_x = x^{\alpha-1} e^{\lambda(T-t)} + \varepsilon^2 H_x + \varepsilon^4 G_x,$$

$$Q_y = \varepsilon^4 G_y,$$

$$Q_{xx} = (\alpha-1) x^{\alpha-2} e^{\lambda(T-t)} + \varepsilon^2 H_{xx} + \varepsilon^4 G_{xx},$$

$$Q_{xy} = \varepsilon^4 G_{xy},$$

$$Q_{yy} = \varepsilon^4 G_{yy}.$$

Now we do the same as in the previous chapter. As in the previous case $O(1)$ term trivially gives us nothing so we'll deal with $O(\varepsilon^2)$ term, which after some cancellations gives us the following equation:

$$\frac{1}{2}\sigma^2 u^2 x^2 (u-1)^2 G_{yy} + \hat{N}(H) + \frac{1}{2}\sigma^2 y^2 (\alpha-1) x^{\alpha-2} e^{\lambda(T-t)} = 0,$$

where by $\hat{N}$ we denote, as usual, the expression with $H(t,x)$. Hence we get almost the same equation as (3.7), but with the small difference

$$M = \frac{1-\alpha}{u^2(1-u)^2} e^{\lambda(T-t)}\ x^{\alpha-4}.$$

With a little work, using some version of the (3.8) and expanding now the free term $\frac{x^\alpha}{\alpha}$ in Tailor series, one can obtain that

$$\hat{Y}^+ = \sqrt[4]{\frac{12\widetilde{r}}{M}} \text{ with } \widetilde{r} = r e^{\lambda(T-t)} x^{\alpha-1}.$$

Hence, finally we get:

$$\hat{Y}^+ = \sqrt[4]{\frac{12 r u^2 (1-u)^2}{1-\alpha}}\ x^{\frac{3}{4}}.$$

So we've got the same solution as in the power form case.

Exponential form of the value function

Let us try another form of the value function:

$$Q(t,x,y) = e^{\left(\ln \frac{x^{\alpha}}{\alpha} + \lambda(T-t) + \varepsilon^2 H(t,x) + \varepsilon^4 G(t,x,y) + ..\right)}.$$

All the derivatives are as follows (after cancellation of the multiplier Q):

$$Q_t = -\lambda + \varepsilon^2 H_t + \varepsilon^4 G_t,$$

$$Q_x = \frac{\alpha}{x} + \varepsilon^2 H_x + \varepsilon^4 G_x,$$

$$Q_y = \varepsilon^4 G_y,$$

$$Q_{xx} = \left(\frac{\alpha}{x} + \varepsilon^2 H_x + \varepsilon^4 G_x\right)^2 - \frac{\alpha}{x^2} + \varepsilon^2 H_{xx} + \varepsilon^4 G_{xx},$$

$$Q_{xy} = \varepsilon^4 G_y(...) + \varepsilon^4 G_{xy},$$

$$Q_{yy} = \varepsilon^4 G_{yy} + \varepsilon^8 (G_y)^2.$$

(...)-term means that we don't need the exact value here. Now we do the same work as in the previous cases and after a little work we get

$$\frac{1}{2}\sigma^2 u^2 x^2 (u-1)^2 G_{yy} + \hat{N}(H) + \frac{1}{2}\sigma^2 y^2 \frac{\alpha(\alpha-1)}{x^2} = 0,$$

where by $\hat{N}$ we denote, as usual, the expression with $H(t,x)$. Hence we get almost the same equation as (3.7) but with the following differences

$$M = \frac{\alpha(1-\alpha)}{u^2(1-u)^2}\, x^{-4}$$

and

$$\hat{Y}^+ = \sqrt[4]{\frac{12\tilde{r}}{M}} \text{ with } \tilde{r} = r\frac{\alpha}{x}.$$

So, finally we obtain:

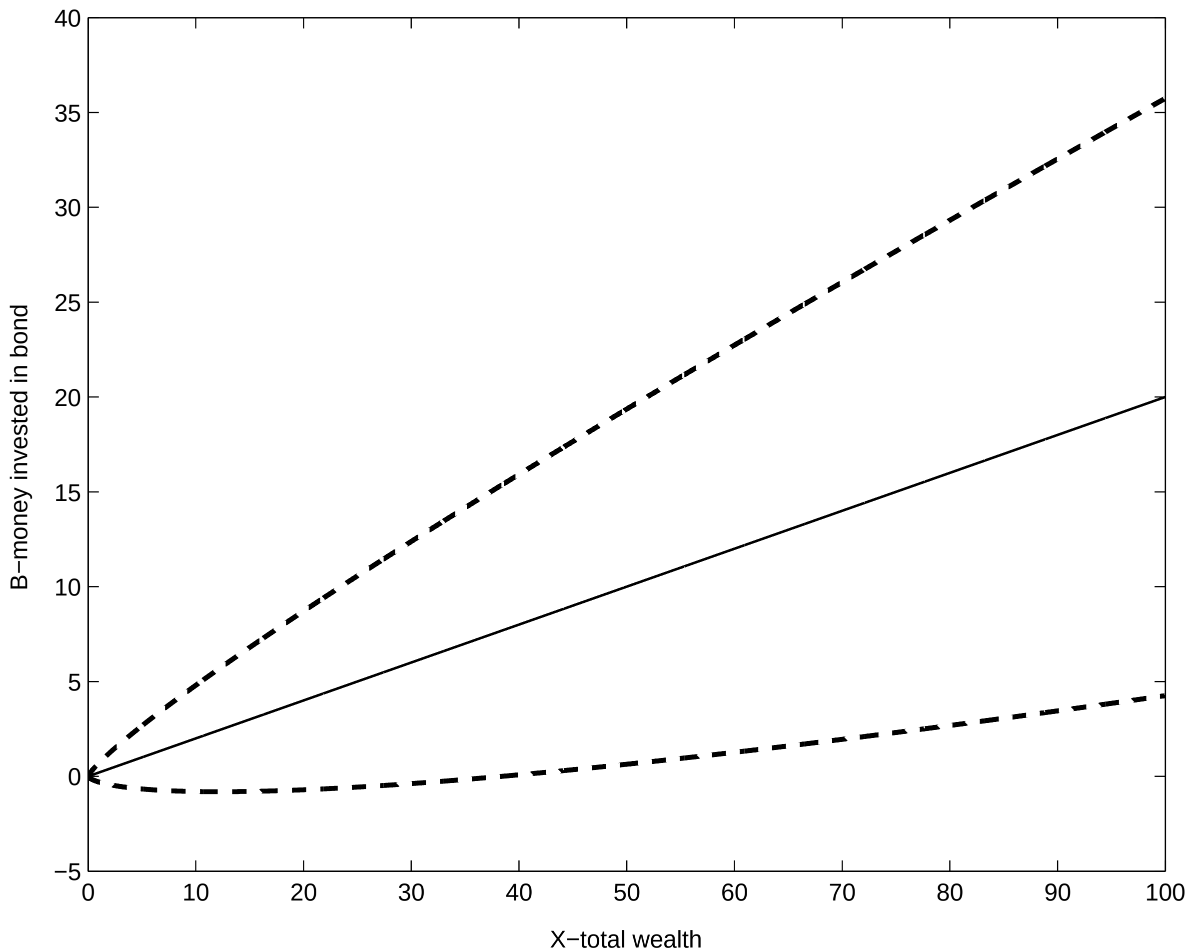
$$\hat{Y}^+ = \sqrt[4]{\frac{12ru^2(1-u)^2}{1-\alpha}}\, x^{\frac{3}{4}}.$$

Notice that we've got the same solution as in the power and linear form cases.

The power expansion chosen in the previous subsection is clearly the most friendly one since it allows us to separate the transaction cost k from non-epsilon term without Tailor series expansion. But it's obvious that never mind what kind of expansion we've chosen, the solution must be the same (under the assumption that ε^5 is negligible, of course).

For the parameters ($\varepsilon = 0.1; \alpha = 0.5; \mu = 0.4; \sigma = 1; r = 1000$) the optimal boundaries in the (X, B)- plane look like (the dotted lines are the boundaries and the bold one is the *Merton line*.)

Figure 3.2: The optimal boundaries



3.4 Checking the optimality condition

First let's verify that $Q > MQ$ in the no-transaction region, i.e. check that

$$Q(t, x, y) > Q(t, x - k, Y^+) \ \ \forall y \in (0, \widehat{Y}^+) \tag{3.16}$$

w.l.o.g. we consider the upper bound only. From the expansion of the function Q in (3.6) and the fact that $Y^+ = 0$ the above condition (3.16) turns to be

$$x + \varepsilon^4 G(t, x, y) > x - \varepsilon^4 r + \varepsilon^4 G(t, x - k, 0),$$

which implies

$$\frac{1}{12} M y^4 + \frac{1}{2} N y^2 + r > 0.$$

Expressing $N = -\frac{1}{3} M (\widehat{Y}^+)^2$ from (3.14) it remains to prove that:

$$\frac{1}{12} M y^2 \Big(2(\widehat{Y}^+)^2 - y^2 \Big) < r, \quad \text{for } \ 0 < y < \widehat{Y}^+$$

or, which is just the same,

$$y^2 \Big(2(\widehat{Y}^+)^2 - y^2 \Big) < \frac{12r}{M}, \quad \text{for } \ 0 < y < \widehat{Y}^+.$$

This is simply checked, since for the function $y^2 \Big(2(\widehat{Y}^+)^2 - y^2 \Big)$ always holds

$$y^2 \Big(2(\widehat{Y}^+)^2 - y^2 \Big) \leq \frac{4(\widehat{Y}^+)^4}{4} = (\widehat{Y}^+)^4 = \frac{12r}{M}$$

and the maximum is achieved only at $y = \widehat{Y}^+$, which fully agrees with (3.16). Since $\widehat{Y}^+ = -\widehat{Y}^-$ the same is proved for the lower bound.

3.5 Discussing the accuracy of approximation

One may wonder - the policy that we have obtained here is stationary, i.e. it doesn't depend on t. We have to say that such policy is obtained only because of the approximation.

From (3.11) in the derivation of the optimal policy we were able to express the possible t-dependent value N as a function of M only, which was the function of x.

This surely affects some regularity condition of the value function, i.e. the boundary condition $G(T, x, y) = 0$ can no longer be achieved, since in the expression of G

$$G(t, x, y) = \frac{1}{12}My^4 - \frac{1}{6}M(\widehat{Y}^+)^2y^2 + D$$

only the last term D can depend on t. But from the previous subsection we know that

$$-r \leq \frac{1}{12}My^4 - \frac{1}{6}M(\widehat{Y}^+)^2y^2 \leq 0,$$

and even by making D not t-dependent, the error is of order $\varepsilon^4 r$ which is of order of one transaction payment. It means that we have actually found the strategy for optimizing utility with utility function equal $\frac{x^\alpha}{\alpha} + \varepsilon^4 f(x)$ for some function f, possibly constant, instead of ordinary power utility $\frac{x^\alpha}{\alpha}$.

Note that the same boundary condition for function H is not affected since we can solve the differential equation $N = -\frac{1}{3}M(\widehat{Y}^+)^2$ and apply the condition $H(T, x) = 0$.

Another boundary condition that we should look at is the behavior of the function Q in the neighborhood of zero, i.e. when x goes to zero. First note that we have obtained the G function in the NT region, i.e. $y \leq \widehat{Y}^+ \propto x^{\frac{3}{4}}$. It means that $x \to 0$ implies $y \to \widehat{Y}^+ \propto x^{\frac{3}{4}}$. Under such conditions we have already seen that $G \to -r + D$. Recall that G is the ε^4 - term. Now looking at the non-epsilon term x in the expression of Q (3.6) and assuming x goes to zero and in particular to $k = \varepsilon^4 r$ we see that there are no problems here and one has only to define exactly how the process must behave in such situation and to define boundary conditions on H and D.

3.6 Maximizing logarithmic utility. Kelly criterion

In this subsection we use methods described in the previous chapters to solve the so called Kelly's criterion problem for another utility from the CRRA class. Now the problem is to find

$$w(t,x) = \sup_{\mathcal{A}} E\Big[\ln X(T) \mid X(t) = x\Big].$$

The problem and it's solution in the no-transaction costs case are very similar to those of power function that we have discussed so far.

In the absence of transaction costs, the optimal solution is always to keep the same proportion $u = \frac{\mu}{\sigma^2}$ of the total wealth in stocks and the value function is given as

$$w(t,x) = \ln x + \lambda(T-t), \quad \text{where} \quad \lambda = \frac{\mu^2}{2\sigma^2}.$$

So we assume the following form of the value function

$$Q(t,x,y) = \ln x + \lambda(T-t) + \varepsilon^2 H(t,x) + \varepsilon^4 G(t,x,y) + ..$$

The derivatives are given as:

$$Q_t = -\lambda + \varepsilon^2 H_t + \varepsilon^4 G_t,$$

$$Q_x = \frac{1}{x} + \varepsilon^2 H_x + \varepsilon^4 G_x,$$

$$Q_y = \varepsilon^4 G_y,$$

$$Q_{xx} = -\frac{1}{x^2} + \varepsilon^2 H_{xx} + \varepsilon^4 G_{xx},$$

$$Q_{xy} = \varepsilon^4 G_{xy},$$

$$Q_{yy} = \varepsilon^4 G_{yy}.$$

Looking at the $O(\varepsilon^2)$-term, after a little work we get:

$$\frac{1}{2}\sigma^2 u^2 x^2 \Big(H_{xx} + (u-1)^2 G_{yy}\Big) - \frac{1}{2}\sigma^2 y^2 \frac{1}{x^2} + \mu u x H_x + H_t = 0.$$

This is rather familiar expression from which we derive the equation similar to (3.7) but with some minor changes

$$M = \frac{1}{u^2(1-u)^2}\, x^{-4}$$

and

$$\hat{Y}^{+} = \sqrt[4]{\frac{12\widetilde{r}}{M}} \text{ with } \widetilde{r} = \frac{r}{x}.$$

So, we get :

$$\hat{Y}^{+} = \sqrt[4]{12ru^2(1-u)^2}\ \ x^{\frac{3}{4}}.$$

Notice that this policy is the limit of power utility policies as $\alpha \downarrow 0$.

3.7 Different approximation approach

Here we discuss a different policy, not optimal for our utility functions, but which is optimal in many other problems (see [26] for example) and which has some remarkable features - we can say much more about the distribution of the total wealth under such policy. It can also be applied as an approximation for the policy that we've obtained in this work.

We are talking about the policy with linear boundaries, i.e. the case when upper and lower optimal boundaries for the bond process B are given as:

$$\widehat{B}^{+} = (1-u)X + \varepsilon hX \quad \text{and} \quad \widehat{B}^{-} = (1-u)X - \varepsilon hX$$

for some constant positive h. Hence the NT -*no transaction* region for the bond process is given as

$$(1-u)X - \varepsilon hX \leq B \leq (1-u)X + \varepsilon hX,$$

i.e. the optimal policy is to trade at any instance the bond reaches the boundaries and to move the bond to the optimal restarting line $B^{*} = (1-u)X$.

This is equivalent to the fact that the NT region for the stock S is given as

$$uX + \varepsilon hX \leq S \leq uX - \varepsilon hX \quad \text{or} \quad \widehat{S}^{-} < S < \widehat{S}^{+},$$

with $\widehat{S}^{+} = uX - \varepsilon hX$ and $\widehat{S}^{-} = uX + \varepsilon hX$.

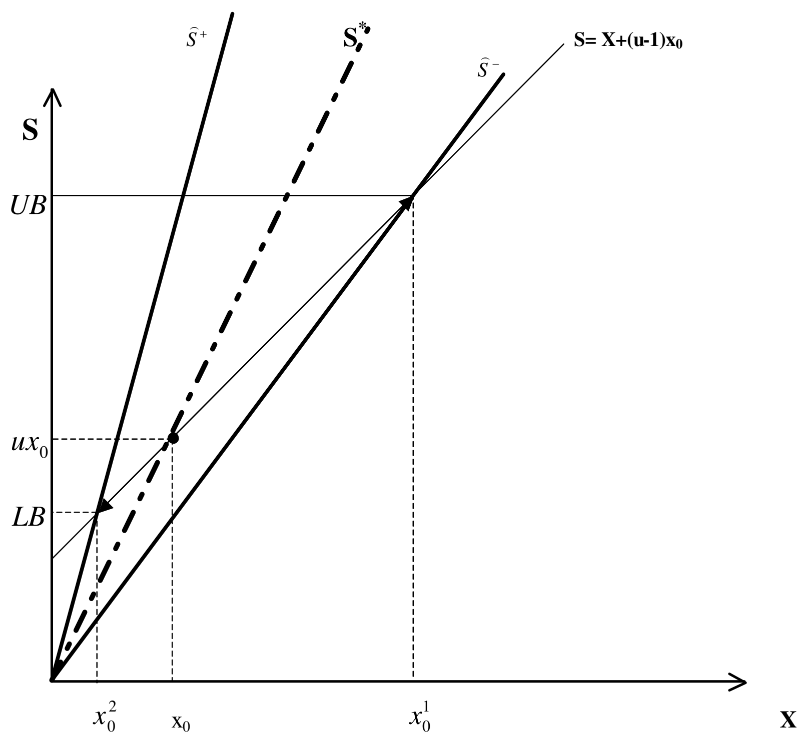
Such a policy can be described in the following way:

Assume w.l.o.g. that investor starts the process when the portfolio is rebalanced in the optimal way: the optimal amount of money in stocks is $S^* = uX$, this is the central line in the (X, S) plane. For instance if the investor starts with the initial wealth x_0, then his money in bond is $B^* = (1-u)x_0$ and in stocks are $S^* = ux_0$.

Since the dependence of S in X is linear : $S = X - B$, then in the (X, S) plane the dependence of S in X in the no-transaction region, can be described as (see the picture)

$$S = X - (1-u)x_0 = X + (u-1)x_0.$$

Figure 3.3: Behavior of the process under the linear strategy



So the first time $S = X + (u-1)x_0$ reaches one of the boundaries $uX - \varepsilon hX$ or $uX + \varepsilon hX$ the investor has to rebalance his portfolio: his new wealth is reduced to to $X-k$ because of transaction payment and stock is brought once again to optimal proportion.

If the initial wealth is x_0 we define the two possible crossing points : x_0^1 for upper and x_0^2 for lower. It means that the stocks process S which behaves like geometric Brownian motion and starts at ux_0 is always situated between the boundaries $ux_0^1 - \varepsilon h x_0^1$ and $ux_0^2 + \varepsilon h x_0^2$.

The first time S exits from this region - the transaction takes place. These crossing points are easily calculated for the linear boundaries case:

$$x_0^1 = \frac{u-1}{u-1-\varepsilon h}\, x_0, \quad x_0^2 = \frac{u-1}{u-1+\varepsilon h}\, x_0.$$

and as a result the interval for S is given as

$$(ux_0 - \frac{\varepsilon h}{u-1+\varepsilon h} x_0 \ ; \ ux_0 + \frac{\varepsilon h}{u-1-\varepsilon h} x_0).$$

We want to calculate the expected exit time of the Geometric Brownian Motion from the above interval starting at ux_0. The problem can be rephrased as finding the exit time of a Brownian motion with drift

$$(\frac{\mu}{\sigma} - \frac{\sigma}{2})t + W_t = \nu t + W_t, \quad \text{where} \ \ \nu = \frac{\mu}{\sigma} - \frac{\sigma}{2},$$

which starts at zero, from the interval

$$(A \ ; \ B) = \Big(\frac{1}{\sigma} \ln{(1 - \frac{\varepsilon h}{u^2 - u + \varepsilon h u})} \ ; \ \frac{1}{\sigma} \ln{(1 + \frac{\varepsilon h}{u^2 - u - \varepsilon h u})} \Big).$$

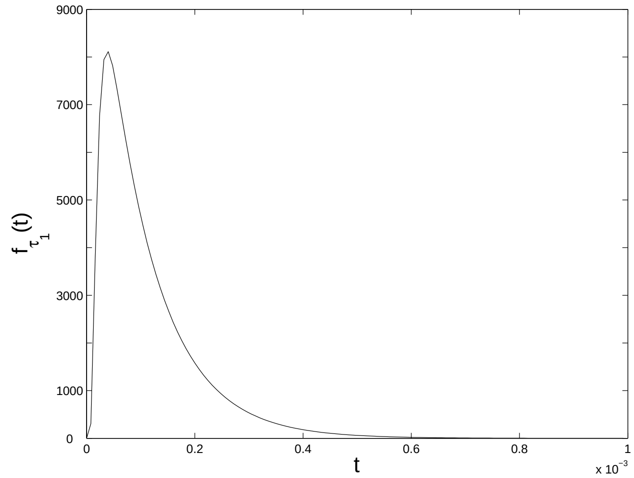
Surprisingly we note that the dependence in x_0 disappears. This is the feature that makes such a policy so interesting. Unfortunately this can be observed only in the linear boundaries case.

So if we denote by τ the time from the beginning up to the first transaction, we can write down explicitly the Laplace transform of τ_1 (see [4])

$$E^0\Big(e^{-\gamma\tau_1}\Big) = \frac{e^{\nu A}\sinh(B\sqrt{2\gamma+\nu^2}) + e^{\nu B}\sinh(-A\sqrt{2\gamma+\nu^2})}{\sinh((B-A)\sqrt{2\gamma+\nu^2})}.$$

and its density function $f_{\tau_1}(t)$ can be calculated as well.

Figure 3.4: The density function of τ_1



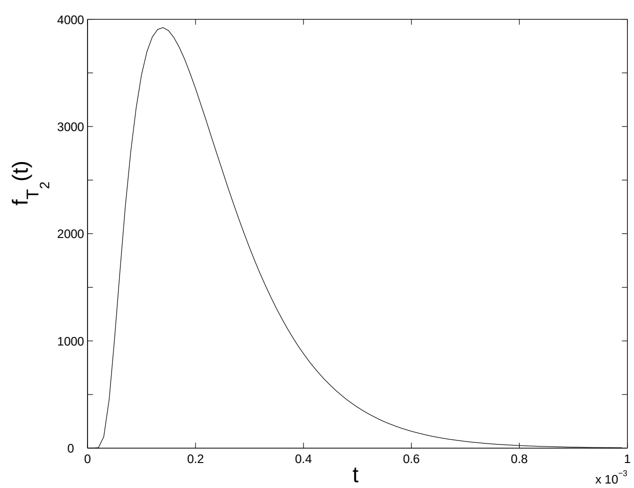
the parameters taken are $\mu = 0.6$, $\sigma = 1$, $h = 2$, $\varepsilon = 0.001$, $u = 0.75$.

Because of independence of this formula in x_0, the independent times between transactions are also identically distributed, so we can calculate the Laplace transform of $T_n = \tau_1 + \tau_2 + ... + \tau_n$ - the time of n transactions

$$E^0\Big(e^{-\gamma T_n}\Big) = E^0\Big(e^{-\gamma(\tau_1+\tau_2+...+\tau_n)}\Big) = \Big[E^0\Big(e^{-\gamma\tau_1}\Big)\Big]^n.$$

So we know the distribution of T_n. For instance look at $f_{T_2}(t)$ - the density function of T_2 (all the parameters are the same):

Figure 3.5: The density function of T_2

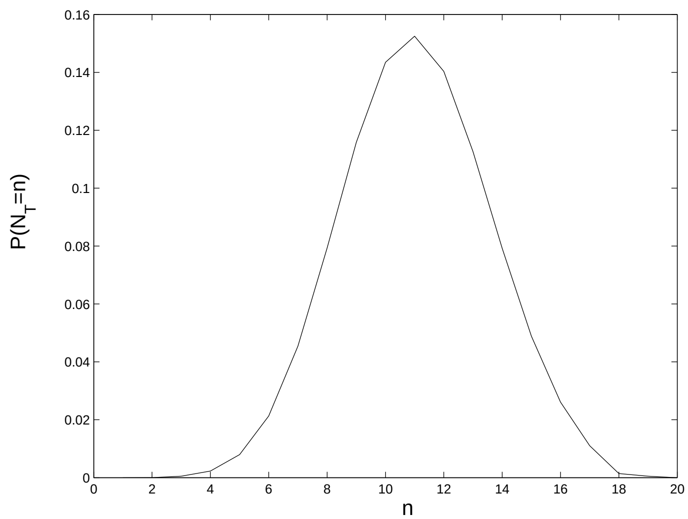


With the help of it we can calculate the distribution of N_t-number of transactions from the beginning up to time t, by the following relation

$$P(N_t = n) = P(T_n < t) - P(T_{n+1} < t).$$

Look at the example with the same parameters as before and $t = 0.0012$.

Figure 3.6: The distribution of N_t



So conditioning on N_T - the total number of transactions and forgetting for a moment about transaction costs, we can obtain the distribution of the wealth process X_t, which starts at x_0 and under such policy behaves like *binary* random variable.

Denoting by $X_n = X_{T_n}$, we know

$$P(X_n = x_0 \alpha_{up}^k \, \alpha_{down}^{n-k}) = \frac{n!}{k!(n-k)!} p^k q^{n-k},$$

where

$$\alpha_{up} = 1 + \frac{\varepsilon h}{u - 1 - \varepsilon h}, \qquad \alpha_{down} = 1 - \frac{\varepsilon h}{u - 1 + \varepsilon h}$$

and the probabilities of going "up" and "down" are given as

$$p = e^{\nu B} \frac{\sinh(-A|\nu|)}{\sinh((B-A)|\nu|)} \;\; ; q = e^{\nu A} \frac{\sinh(B|\nu|)}{\sinh((B-A)|\nu|)}.$$

Now it's up to the reader to decide in which form to add transaction costs in order to keep the above calculations still relevant.

We can add some words on estimation of the expected time between transactions in the linear boundary case.

In this case we are simply driven to the problem of estimating the expectation of the first exit time of time homogeneous diffusion $S(t)$ from the interval $(x - \varepsilon x,\ x + \varepsilon x)$, given that it starts at x. But since in our case the result doesn't depend on x, we can look at the interval $(1 - \varepsilon,\ 1 + \varepsilon)$ for the $S(t)$ which starts at 1. Or we can look at the interval $(x - \varepsilon,\ x + \varepsilon)$ and then take $x = 1$. It is well-known that if we take an interval $[a, b]$, some diffusion process $S(t)$ which starts at $S(t_0) = x \in (a, b)$ then $V(x) = E^x(\tau)$, with $\tau = \inf\{t > t_0 \mid S(t) \notin (a, b)\}$ is given as a solution of an ordinary differential equation

$$Lv = -1 \quad \text{with } v(a) = v(b) = 0.$$

In our case it turns to be

$$\frac{1}{2}\sigma^2 x^2 v^{''}(x) + \mu x v^{'}(x) = -1, \text{ with } v(x + \varepsilon) = v(x - \varepsilon) = 0.$$

Solving the equation, and denoting $\gamma = \frac{2\mu}{\sigma^2}$ we get that

$$v(x) = \frac{2}{\sigma^2 \gamma}\Big[\ln\Big(1 + \frac{\varepsilon}{x - \varepsilon}\Big) + \frac{x^{1-\gamma} - (x - \varepsilon)^{1-\gamma}}{(x - \varepsilon)^{1-\gamma} - (x + \varepsilon)^{1-\gamma}} \ln\Big(1 + \frac{2\varepsilon}{x - \varepsilon}\Big)\Big].$$

Denoting by J the big fraction there, we see that

$$J = \frac{x^{1-\gamma} - (x - \varepsilon)^{1-\gamma}}{(x - \varepsilon)^{1-\gamma} - (x + \varepsilon)^{1-\gamma}} =$$

$$-\frac{\frac{x^{1-\gamma} - (x-\varepsilon)^{1-\gamma}}{\varepsilon}}{\frac{(x+\varepsilon)^{1-\gamma} - x^{1-\gamma}}{\varepsilon} + \frac{x^{1-\gamma} - (x-\varepsilon)^{1-\gamma}}{\varepsilon}} \simeq -\frac{1}{2} + o(\varepsilon).$$

So returning to $v(x)$ and writing ε instead of $\frac{\varepsilon}{x - \varepsilon}$ we get

$$v(x) \simeq \varepsilon - \frac{\epsilon^2}{2} - \frac{2\varepsilon}{2} + \frac{4\varepsilon^2}{2} + ... = O(\varepsilon^2).$$

Coefficient optimization in the *linear - boundary* policy

In reality there are at least two main deficiencies in the above analysis. First, the only thing that we know analytically is the expression for the Laplace transform of the density of T_n - the time of n transactions. Any other quantities, such as distribution of N_t - the total number of interventions, and more importantly the expectation of the utility of X_T, can be calculated only numerically. Second, it is worth to note that the binary tree for X_t is not recombining once transaction costs are introduced. It means that at each intervention we have to subtract a transaction payment from the current wealth and only after that multiply it by some constant.

All this makes any analytical analysis and optimization inapplicable. Hence any optimization that can be done is via simulations only. Here we present some results.

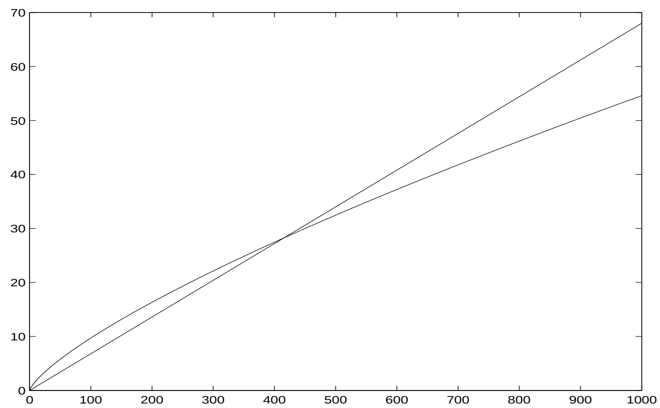
The parameters taken are

1. initial wealth $x = 100$,
2. $\varepsilon = 0.04$ and $r = 10000$, so the fixed transaction cost is $k = r\varepsilon^4 = 0.00256$
3. the process parameters are $\mu = 0.6$, $\sigma = 1$, $\alpha = 0.2$.
4. the initial time is $t_0 = 0$ and the terminal $T = 0.1$, the time step dt is taken to be $dt = 0.001$, so the number of steps is 100.
5. number of iterations is $N = 10000$

We are testing the *linear - boundary* policy, i.e. the policy of the kind $\hat{Y}^+ = \varepsilon h x$. By changing the parameter h for each simulation, it's possible to obtain the value of h, which optimizes the performance of the system.

In the current case the optimal h_{opt} turned out to be about 1.6. In the figure below we compare the best linear boundary with the optimal one according to our analytical analysis:

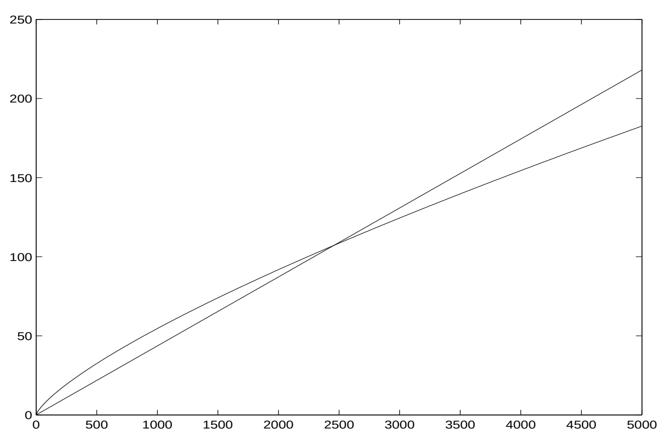
Figure 3.7: Optimizing linear boundaries (initial wealth =100)



So the similarity is obvious. It is worth noting that the slope of the linear approximation is close to the slope of the optimal curve around x=100.

Clearly, the results of such optimization have to depend on the initial wealth x_0. Consequently we repeat the scheme with initial wealth of 1000 rather than 100. We expect the optimal h to be smaller, and this is indeed the case: $h_{opt} = 1.2$. The plot now looks like:

Figure 3.8: Optimizing linear boundaries (initial wealth =1000)



and once again the slope of the linear approximation attempts to approximate our optimal curve, this time at the vicinity of x=1000.

4 Goal Problem

Another kind of problems, arising in the context of portfolio optimization are *goal problems*. In this kind of problems the investor starting at time t, wishes to maximize the probability that the total wealth reaches a specified level by some terminal time T.

4.1 Goal Problem in the no-transaction costs case

4.1.1 The classic approach

Denote, like in previous chapters, the investor's wealth by $X(t) = B(t) + S(t)$. Clearly under the assumption that $\sigma = 1$ the wealth process solves

$$dX(s) = S(s)(\mu ds + dW_s), \quad t \leq s \leq T.$$

Our admissible policies $S(s)$ are taken to be $\mathcal{F}_s$-measurable, satisfying almost surely the integrability condition $\int_t^T S(s)^2 ds < +\infty$ and the state constraint $0 \leq X_t \leq 1$.
Define by $\mathcal{A}$ the set of all admissible policies.

The objective is to avoid absorption at the origin and to maximize the probability of reaching the goal $X = 1$ by the expiration time T. In other words, the value function is given by

$$w(t, x) = \sup_{\mathcal{A}} P\Big(X(T) = 1 \mid X(t) = x\Big).$$

The absence of transaction costs the solution was first derived using martingale methods by Heath [10] , Karatzas [11] and is as follows:

$$w(t, x) = \Phi\Big(\Phi^{-1}(x) + \mu\sqrt{T - t}\Big).$$

The optimal strategy says that the amount of money held in stock at time t has to be

$$S^*(t) = \frac{1}{\sqrt{T - t}} \varphi\Big(\Phi^{-1}(X(t))\Big).$$

where $\Phi(\cdot)$ is the standard normal distribution function and $\varphi(\cdot)$ is the density of the standard normal distribution.

Note that the solution is obtained without HJB equations, but it still can be viewed as viscosity solution of the corresponding HJB:

$$\begin{cases} w_t + \sup_S \left[\frac{1}{2}S(t)^2 w_{xx} + \mu S(t) w_x\right] = 0, \\ w(x,T) = \begin{cases} 0, & 0 \le x < 1, \\ 1, & x = 1. \end{cases} \end{cases} \tag{4.1}$$

4.1.2 A combinatorial approach.

We solve the Goal Problem in the no-transaction case via Markov Chain approximation method. Assume that we start at time $t = 0$ with initial capital $x < 1$ and the aim is to get to 1 up to time T. Divide the time interval into n equal intervals $dt = \frac{T}{n}$.

The behavior of the stock money

$$dS(t) = S(t)(\mu dt + dW_t)$$

can be approximated by the model such that given the value $S(t)$ at time t, the change $dS(t)$ can obtain values

$$dS(t) = \begin{cases} \sqrt{dt}S(t) & w.p.\ \ p = \frac{1}{2} + \frac{\mu}{2}\sqrt{dt} \\ -\sqrt{dt}S(t) & w.p.\ \ q = \frac{1}{2} - \frac{\mu}{2}\sqrt{dt} \end{cases}$$

where $\mu > 0$ implies $p > q$.

So investor can trade only n times (on $t = 0, \frac{T}{n}, \frac{2T}{n}, .., \frac{n-1}{n}T$). What we have now is a discrete-time optimization problem of finding $w_n(x)$ - the maximal probability of reaching 1, under the assumption that the initial capital at $t = 0$ is x and there are n time-intervals.

The fact that discrete-time behavior of risky money is symmetric, i.e. without looking at probabilities, the investor can lose and win exactly the same amounts of money, makes it possible to simplify the problem. Note also that if the investor reaches zero, then the game is stopped and he can't continue, so he's not allowed to make bets which can lead him to zero with positive probability.

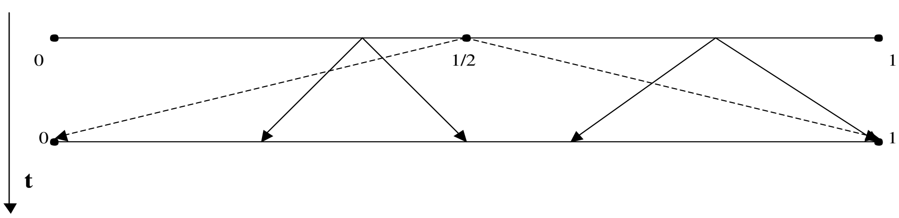
Go backwards in time. For $n = 0$ investor has no possibility to trade and his optimal value function w_0 clearly equals zero for all x except 1, where it surely equals 1. i.e. $w_0(x) = 1_{\{1\}}(x)$.

Now for all $n \geq 1$ divide x-axis into 2^n intervals

$$I_j^n = [\frac{j}{2^n}; \frac{j+1}{2^n}) \ , \ 0 \leq j \leq 2^n - 1, \ \text{ assume for convention that } \ I_{2^n}^n = \{1\}.$$

Look at $n = 1$:

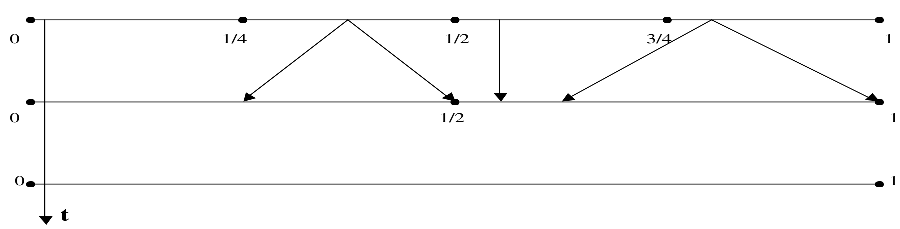
Figure 4.1: Simple trading model (n=1)



If the investor starts at $x \in [0; \frac{1}{2})$ then the probability of reaching 1 in *just one step* is 0, because the only way he can behave is to bet a sum of money which is enough to reach 1, but looking at his position it's clear that this bet can lead to bankruptcy if he loses, so he's not allowed to make such steps. On the other hand, if he starts at the interval $[\frac{1}{2}; 1)$, then the optimal policy for him is to try to reach 1 with probability p. So $w_1(x) = 0$ for all $x \in [0; \frac{1}{2})$ and $w_1(x) = p$ for $x \in [\frac{1}{2}; 1)$. Clearly $w_1(1) = 1$.

Now make one step backwards as in dynamic programming and look at $n = 2$:

Figure 4.2: Simple trading model (n=2)



If the initial x belongs to $[0; \frac{1}{4})$ then the probability of reaching 1 in at most two steps equals zero, because from this interval it's impossible to achieve 1 without betting "illegal" sums of money - i.e. sums which may lead to bankruptcy. Now if $x \in [\frac{1}{4}; \frac{1}{2})$ then there's a probability p of reaching the interval $[\frac{1}{2}; 1)$ on the next step and from where the optimal probability is known p. So the maximal value function for this interval equals p^2. Doing the same work for the two remaining intervals, we have that $w_2(x) = p = p^2 + pq$ for $x \in [\frac{1}{2}; \frac{3}{4})$ and $w_2(x) = p + pq = p^2 + 2pq$ for $x \in [\frac{3}{4}; 1)$.

First of all it seems natural to assume that the maximal probability of reaching the goal under some *optimal strategy* is the same for all x from each such interval. Proceeding this way with the help of *computer* we've found the maximal probability function $w_n(x)$ for $n < 16$ as a piecewise linear function which is constant on each interval. What was really interesting - was the fact that the differences between values on intervals were behaving like the jumps of the *Binom of Newton*- in decreasing order because of $p > q$. For instance taking $n = 3$ ($2^3 = 8$ intervals), it can be seen as

Figure 4.3: The value function example

interval	The value $w_3(x)$	The difference
$[0, \frac{1}{8})$	0	
$[\frac{1}{8}, \frac{2}{8})$	p^3	p^3
$[\frac{2}{8}, \frac{3}{8})$	$p^3 + p^2q$	p^2q
$[\frac{3}{8}, \frac{4}{8})$	$p^3 + 2p^2q$	p^2q
$[\frac{4}{8}, \frac{5}{8})$	$p^3 + 3p^2q$	p^2q
$[\frac{5}{8}, \frac{6}{8})$	$p^3 + 3p^2q + pq^2$	pq^2
$[\frac{6}{8}, \frac{7}{8})$	$p^3 + 3p^2q + 2pq^2$	pq^2
$[\frac{7}{8}, 1)$	$p^3 + 3p^2q + 3pq^2$	pq^2
$\{1\}$	1	q^3

This fact can be proved by induction for all n (at step n all such differences (or jumps) are decreasing as x approaches 1 and each jump d at step n splits into two jumps pd and qd at the next step $n + 1$ and so on..), so we will not provide the full proof here.

Now the way to find the limit of such value function as $n \to \infty$ is using central limit theorem. For n large enough we can write that for all x there exist some number $k(x)$, such that

$$w_n(x) = \sum_{i=0}^{k(x)} \frac{n!}{i!(n-i)!} p^{n-i} q^i =$$

$$= \sum_{i=0}^{k(x)} \frac{n!}{i!(n-i)!} \Big(\frac{1}{2} + \frac{\mu}{2}\sqrt{dt}\Big)^{n-i} \Big(\frac{1}{2} - \frac{\mu}{2}\sqrt{dt}\Big)^{i}, \tag{4.2}$$

and this $k(x)$ is characterized by $\sum_{i=0}^{k(x)} \frac{n!}{i!(n-i)!} \left(\frac{1}{2}\right)^n \simeq x$, ,which can be viewed in the limit as

$$P\Big(Y_n < k(x)\Big) = x \;\text{ with }\; Y_n \sim N(\frac{n}{2};\;\; \frac{n}{4}),$$

where using CLT,

$$k(x) \simeq \Phi^{-1}(x)\frac{\sqrt{n}}{2} + \frac{n}{2}. \tag{4.3}$$

The equality (4.2) can be viewed as

$$w_n(x) = P\Big(Z_n < k(x)\Big) \;\text{ with }\; Z_n \sim N(nq;\;\; nqp).$$

So we have that

$$w_n(x) = \Phi\Big(\frac{k(x) - \frac{n}{2} + \frac{\mu}{2}\sqrt{dt}}{\frac{\sqrt{n}}{2}\sqrt{1-\mu^2 dt}}\Big).$$

Putting here the expression for $k(x)$ from (4.3) and the fact that $dt = \frac{T}{n}$ we obtain

$$w_n(x) = \Phi\Big(\frac{\Phi^{-1}(x)\frac{\sqrt{n}}{2} + \frac{n}{2} - \frac{n}{2} + \frac{\mu}{2}\sqrt{dt}}{\frac{\sqrt{n}}{2}\sqrt{1-\mu^2 dt}}\Big) =$$

$$= \Phi\Big(\frac{\Phi^{-1}(x) + \mu\sqrt{T}}{\sqrt{1-\mu^2\frac{T}{n}}}\Big) \to \Phi\Big(\Phi^{-1}(x) + \mu\sqrt{T}\Big)$$

as n goes to infinity. And this is exactly the solution of Heath. After that there were two ways, either to view this solution as the solution of the HJB equation of stochastic control (4.1) and to derive the optimal policy from the equation, or to try and to find the optimal policy via combinatorics. The first way is pretty simple, and deriving the solution via the second method was stopped, because we've discovered that the solution was obtained by Kulldorff [15] by some another combinatorial approach.

4.2 Fixed transaction cost case.

We use the same method of QVI for solving the Goal Problem for the case when there are fixed transaction costs. It's a little bit problematic using here the method proposed in the previous section, because of irregularity of the value function even in the no-transactions case. But the fact that it still can be viewed as a viscosity solution gives us a chance to try QVI inequalities in the goal problem.

We use approximation method. Introduce the new variable Y :

$$S(t) = S^*(t) + \frac{\varepsilon Y(t)}{\sqrt{T-t}} = \frac{1}{\sqrt{T-t}}\Big[\varphi\Big(\Phi^{-1}(X(t))\Big) + \varepsilon Y(t)\Big].$$

For further work we'll need dY. We'll calculate it using Ito's formula from

$$Y(t) = \frac{1}{\varepsilon}\Big(\sqrt{T-t}\, S(t) - \varphi(\Phi^{-1}(X(t)))\Big), \tag{4.4}$$

and

$$dS(t) = S(t)(\mu dt + dW_t).$$

First let's make some useful calculations. Note that

$$\varphi'(x) = -x\varphi(x) \quad \text{and} \quad \frac{d}{dx}\Big[\Phi^{-1}(x)\Big] = \frac{1}{\varphi(\Phi^{-1}(x))}.$$

From this follows that

$$\frac{d}{dx}\Big[\varphi(\Phi^{-1}(x))\Big] = \frac{\varphi'(\Phi^{-1}(x))}{\varphi(\Phi^{-1}(x))} = -\Phi^{-1}(x).$$

and

$$\frac{d^2}{dx^2}\Big[\varphi(\Phi^{-1}(x))\Big] = \frac{d}{dx}\Big[-\Phi^{-1}(x)\Big] = -\frac{1}{\varphi(\Phi^{-1}(x))}.$$

Let's calculate the differential of two terms from (4.4) using Ito's formula

$$d\Big[\sqrt{T-t}\, S(t)\Big] = -\frac{S(t)}{2\sqrt{T-t}}\, dt + S(t)\sqrt{T-t}\, (\mu dt + dW_t),$$

$$d\Big[-\varphi(\Phi^{-1}(X(t)))\Big] = \Phi^{-1}(X(t))S(t)\Big(\mu dt + dW_t\Big) + \frac{S(t)^2}{2\varphi(\Phi^{-1}(X(t)))}dt,$$

Finally we can calculate dY :

$$dY(t) = \frac{S(t)}{\varepsilon}\Big[- \frac{1}{2\sqrt{T-t}} + \mu\sqrt{T-t} + \mu\Phi^{-1}(X(t)) - \frac{S(t)}{2\varphi(\Phi^{-1}(X(t)))}\Big]dt$$

$$+\frac{S(t)}{\varepsilon}\Big[\sqrt{T-t} + \Phi^{-1}(X(t))\Big]\ dW_t.$$

This can be rewritten as

$$dY(t) = \frac{S(t)}{\varepsilon\ \sqrt{T-t}}\Big[- 1 + \mu(T-t) + \mu\sqrt{T-t}\Phi^{-1}(X(t)) - \frac{\varepsilon Y(t)}{2\varphi(\Phi^{-1}(X(t)))}\Big]dt$$

$$+\frac{S(t)}{\varepsilon}\Big[\sqrt{T-t} + \Phi^{-1}(X(t))\Big]\ dW_t. \tag{4.5}$$

In order to simplify the expression let's introduce new variables:

$$\begin{cases} V := -1 + \mu(T-t) + \mu\sqrt{T-t}\Phi^{-1}(X(t)) - \frac{\varepsilon Y(t)}{2\varphi(\Phi^{-1}(X(t)))}, \\ U := \sqrt{T-t} + \Phi^{-1}(X(t)), \\ \Theta := \varphi(\Phi^{-1}(X(t))). \end{cases} \tag{4.6}$$

Together with

$$dX(t) = S(t)(\mu dt + dW_t)$$

we can write the generator of the process $\Big(t, X(t), Y(t)\Big)$ and the corresponding QVI equation $LQ = 0$ as

$$Q_t + \frac{\mu}{\sqrt{T-t}}(\Theta + \varepsilon y)\ Q_x + \frac{(\Theta + \varepsilon y)}{\varepsilon(T-t)}VQ_y$$

$$+\frac{(\Theta + \varepsilon y)^2}{(T-t)}\Big[\frac{1}{2}Q_{xx} + \frac{U}{\varepsilon}Q_{xy} + \frac{U^2}{2\varepsilon^2}Q_{yy}\Big] = 0. \tag{4.7}$$

We will solve the problem for the case when transaction costs are small ,i.e. $k = \varepsilon^4 r$ for some fixed positive r .

Assume that the upper boundary of the no transaction region and optimal upper restarting line have the forms

$$\widehat{S}^+ = \frac{1}{\sqrt{T-t}}\Big[\Theta + \varepsilon\widehat{Y}^+\Big],$$

$$S^{+} = \frac{1}{\sqrt{T-t}}\Big[\Theta + \varepsilon Y^{+}\Big],$$

i.e. when the process (X, Y) reaches the upper boundary $(X, \widehat{Y}^{+})$, then we have to make a transaction and to move it back to the no transaction region i.e. to his new restarting line (X, Y^{+}). Also, we assume a similar form for the lower boundary of NT and for the lower restarting line

$$\widehat{S}^{-} = \frac{1}{\sqrt{T-t}}\Big[\Theta + \varepsilon \widehat{Y}^{-}\Big],$$

$$S^{-} = \frac{1}{\sqrt{T-t}}\Big[\Theta + \varepsilon Y^{-}\Big],$$

(the signs of $\widehat{Y}^{-}$and Y^{-} are negative). And surely all Y's may depend on t and on X.

In the NT we expand Q in the following form

$$Q(t,x,Y) = e^{\Big(H_0(t,x)+\varepsilon^2 H(t,x)+\varepsilon^4 G(t,x,Y)+\dots\Big)} \tag{4.8}$$

with H_0 to match the solution in case $\varepsilon = 0$

$$H_0(x,t) = \ln\Big(\Phi(\Phi^{-1}(x) + \mu\sqrt{T-t})\Big).$$

Now, like in Section **3**, in order to obtain all the bounds, we have the following conditions to hold

$$\begin{cases} Q(t,x,y) = Q(t,x-k,Y^{+}), \quad \forall\, y \geq \widehat{Y}^{+}, \\ \\ Q_y(t,x,y) = Q_y(t,x-k,Y^{+}), \quad \forall\, y \geq \widehat{Y}^{+}, \end{cases} \tag{4.9}$$

as well as *optimality of transaction* condition

$$\begin{cases} \frac{d}{d(\Delta_y)} Q(t,x-k,\widehat{Y}^{+} \;-\; \Delta_y)\Big|_{\Delta_y = \widehat{Y}^{+} - Y^{+}} = 0, \\ \\ Q_{yy}(t,x-k,Y^{+}) < 0. \end{cases} \tag{4.10}$$

For simplicity denote $\Phi(\Phi^{-1}(x) + \mu\sqrt{T-t})$ by Λ.

We need partial derivatives (divide by Q):

$$\frac{Q_t}{Q} = -\frac{\mu}{2\sqrt{T-t}} \frac{\varphi(\Phi^{-1}(x) + \mu\sqrt{T-t})}{\Phi(\Phi^{-1}(x) + \mu\sqrt{T-t})} + \varepsilon^2 H_t + ... = -\frac{\mu}{2\sqrt{T-t}} \frac{\varphi(\Lambda)}{\Phi(\Lambda)} + \varepsilon^2 H_t + ..,$$

$$\frac{Q_x}{Q} = \frac{\varphi(\Lambda)}{\Phi(\Lambda)\Theta} + \varepsilon^2 H_x + \varepsilon^4 G_x,$$

$$\frac{Q_y}{Q} = \varepsilon^4 G_y,$$

$$\frac{Q_{xx}}{Q} = \Big(\frac{\varphi(\Lambda)}{\Phi(\Lambda)\Theta} + \varepsilon^2 H_x + \varepsilon^4 G_x)\Big)^2 + (\frac{\varphi(\Lambda)}{\Phi(\Lambda)\Theta})'_x + \varepsilon^2 H_{xx} + \varepsilon^4 G_{xx},$$

$$\frac{Q_{xy}}{Q} = \varepsilon^4 G_y \Big(...\Big) + \varepsilon^4 G_{xy},$$

$$\frac{Q_{yy}}{Q} = \varepsilon^8 G_y^2 + \varepsilon^4 G_{yy},$$

and

$$\Big(\frac{\varphi(\Lambda)}{\Phi(\Lambda)\Theta}\Big)'_x = -\frac{1}{\Theta^2}\frac{\varphi(\Lambda)}{\Phi(\Lambda)}\Big(\mu\sqrt{T-t} + \frac{\varphi(\Lambda)}{\Phi(\Lambda)}\Big).$$

Now we have to put all these derivatives into equation (4.7). By sorting the equation in powers of ε we thus get a number of smaller equations possibly easier to solve.

Also we are going to use the fact that ε in powers larger than 4 are negligible.

$O(1)$ **term** :

$$-\frac{\mu}{2\sqrt{T-t}}\frac{\varphi(\Lambda)}{\Phi(\Lambda)} + \frac{\mu}{\sqrt{T-t}}\frac{\varphi(\Lambda)}{\Phi(\Lambda)}$$
$$+\frac{1}{2(T-t)}\Big[\Big(\frac{\varphi(\Lambda)}{\Phi(\Lambda)}\Big)^2 - \frac{\varphi(\Lambda)}{\Phi(\Lambda)}\Big(\mu\sqrt{T-t} + \frac{\varphi(\Lambda)}{\Phi(\Lambda)}\Big)\Big] = 0.$$

It can be simplified to

$$\frac{\mu}{2} - \frac{1}{2\sqrt{T-t}}(\mu\sqrt{T-t}) = 0,$$

which trivially holds.

Now we look at $O(\varepsilon^2)$ **term**:

$$H_t + \frac{\mu}{\sqrt{T-t}}\Theta H_x + \frac{\Theta^2}{2(T-t)}\Big[2\frac{\varphi(\Lambda)}{\Theta\Phi(\Lambda)}H_x + H_{xx} + U^2 G_{yy}\Big]$$

$$-\frac{y^2}{2\Theta^2(T-t)}\Big[\mu\sqrt{T-t}\frac{\varphi(\Lambda)}{\Phi(\Lambda)}\Big] = 0.$$

or

$$G_{yy} = My^2 + N(H) = 0$$

with the following solution (as a function of y only)

$$G(t,x,y) = \frac{1}{12}My^4 + \frac{1}{2}Ny^2 + Cy + D. \tag{4.11}$$

where

$$M = \frac{\mu}{2\Theta^2\sqrt{T-t}}\frac{\varphi(\Lambda)}{\Phi(\Lambda)}\Big[\frac{\frac{1}{2}\Theta^2 U^2}{(T-t)}\Big]^{-1} = \frac{\mu\sqrt{T-t}}{\Theta^4\, U^2}\frac{\varphi(\Lambda)}{\Phi(\Lambda)}$$

Also we won't need here the exact value of N.

Here unknown D depend on x and t and C=0 as before.

It's difficult to find an explicit form of the value function Q, but using the boundary conditions and one additional condition (optimality of the transaction) we can find all the Y^+, $\widehat{Y}^+$, Y^-, $\widehat{Y}^-$.

The conditions (4.9) and (4.10) lead to

$$G(t,x,Y^+) - G(t,x,y) = \frac{r}{\Theta}\frac{\varphi(\Lambda)}{\Phi(\Lambda)}, \quad \forall y \geq \widehat{Y}^+. \tag{4.12}$$

In the same way the LB condition is given by

$$G(t,x,Y^-) - G(t,x,y) = \frac{r}{\Theta}\frac{\varphi(\Lambda)}{\Phi(\Lambda)}, \quad \forall y \leq \widehat{Y}^-. \tag{4.13}$$

Thus using the form of G we obtain the equation (4.12) turns to be

$$\frac{1}{12}M\Big((Y^+)^4 - (\widehat{Y}^+)^4\Big) + \frac{1}{2}N\Big((Y^+)^2 - (\widehat{Y}^+)^2\Big) = \frac{r}{\Theta}\frac{\varphi(\Lambda)}{\Phi(\Lambda)}. \tag{4.14}$$

Now let's look at optimality of transaction condition (4.10)

$$\frac{d}{d(\Delta_y)}\Big(Q(t, x-k, \widehat{Y}^+ - \Delta_y)\Big)\Big|_{\Delta_y = \widehat{Y}^+ - Y^+} = 0,$$

where we note that $\widehat{Y}^+ = \widehat{Y}^+(x)$, and $Y^+ = Y^+(x-k)$ - are functions of x and $x-k$ respectively. From the above follows that

$$G_y(t, x-k, Y^+(x-k)) = 0 \quad \forall x > k \qquad \Rightarrow G_y(x, Y^+) = 0.$$

From the requirement of the G_y continuity on the upper border, we also get

$$G_y(t, x, \widehat{Y}^+) = 0$$

From these two conditions follows that

$$G_y(t, x, Y^+) - G_y(t, x, \widehat{Y}^+) = 0, \tag{4.15}$$

or

$$\frac{1}{3}M\Big((Y^+)^2 + Y^+(\widehat{Y}^+) + (\widehat{Y}^+)^2\Big) + N = 0. \tag{4.16}$$

Expressing N from (4.16) and putting it into (4.14), we get

$$(\widehat{Y}^+)^3 - \widehat{Y}^+((Y^+)^2 + Y^+(\widehat{Y}^+) + (\widehat{Y}^+)^2) = 0.$$

from this follows that $Y^+ = 0$.

Putting this into (4.14) we obtain

$$\widehat{Y}^+ = \sqrt[4]{\frac{12\frac{r}{\Theta}\frac{\varphi(\Lambda)}{\Phi(\Lambda)}}{M}} = \mu^{-\frac{1}{4}}\Big(12r\varphi^3(\Phi^{-1}(x))[\sqrt{T-t} + \Phi^{-1}(x)]^2\Big)^{\frac{1}{4}}(T-t)^{-\frac{1}{8}}.$$

From the analogous results on the lower border we get

$$\widehat{Y}^= - \widehat{Y}^+ \quad \text{and } Y^- = -Y^+ = 0.$$

So we have obtained the approximated optimal trading strategy for the Goal Problem.

5 Simulation

The results of this work are theoretical, but we can add a couple of simulations in order to demonstrate the performance of the policy. We will consider *only* the power utility case.

Clearly, with the help of simulations one cannot guarantee that the policy proposed here is indeed optimal, but we'll try to compare our *optimal* policy against some non-trivial policies (denoted by $P1$ and $P2$).

In the simulation we want to model the wealth process behavior from the beginning t_0 up to the terminal time T. The time axis is divided into $\frac{T-t_0}{dt}$ time steps and at each step we generate the stock price change and then rebalance the portfolio according to specific policy (ours or some other). In this way we continue up to the end and after that we calculate the utility function value. We repeat the simulation for some big number N of iterations and then take an average. Thus we get the expected utility.

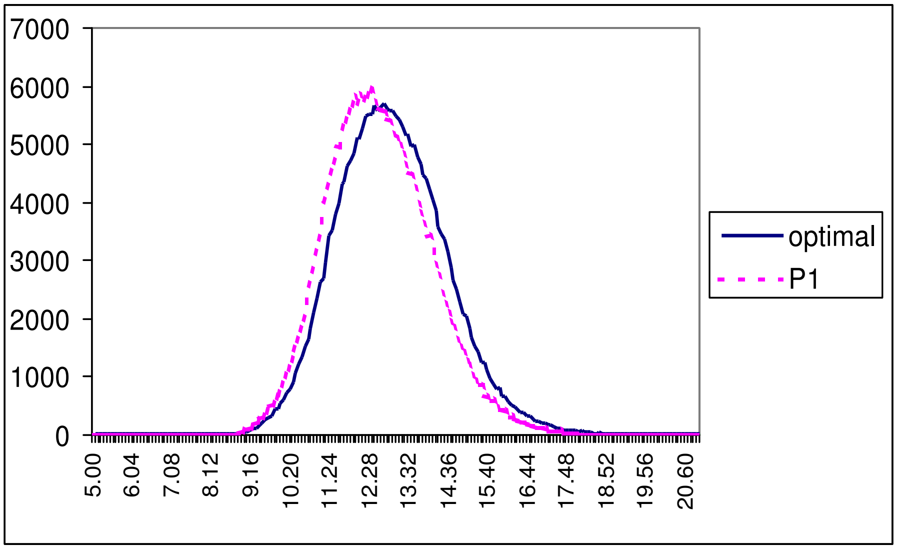
In addition we plot the histogram of the results obtained in order to show that number of iterations is actually enough and the distribution of the values can tell us what policy is better.

In the first simulation we've taken the following parameters:

1. initial wealth $x = 100$,
2. $\varepsilon = 0.02$ and $r = 1000$, so the real transaction cost is $k = r\varepsilon^4 = 0.0016$
3. the initial time is $t_0 = 0$ and the terminal $T = 2$, the time step dt is taken to be $dt = 0.001$, so the number of steps is 2000.
4. the process parameters are $\mu = 0.4$, $\sigma = 1$, $\alpha = 0.2$.

The policy $P1$ is taken as a policy, according to which the number of interventions is equal to the average number of interventions of our *optimal* policy. It means that if in our policy the average number of interventions is 20, then, according to $P1$, the investor has to intervene every $\frac{T-t_0}{20}$ units of time.

Figure 5.1: Comparing two policies (P1)



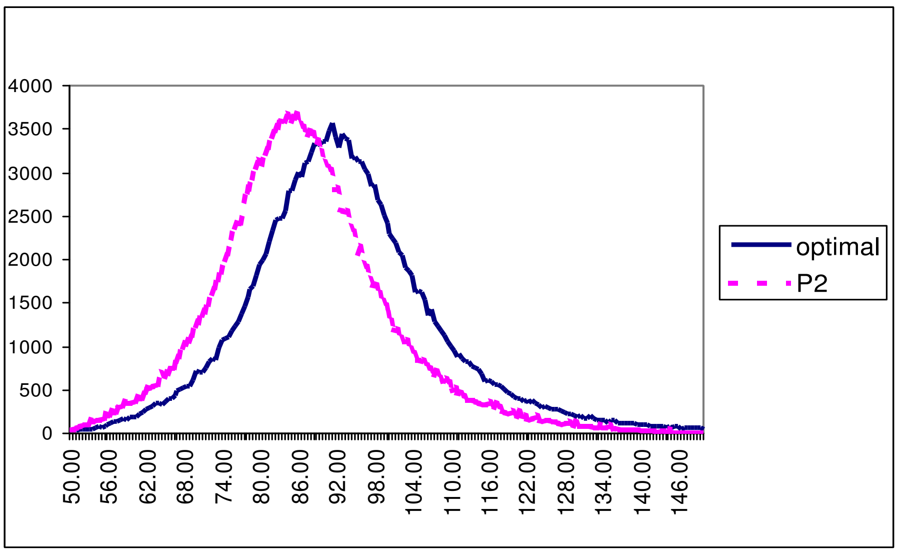
As a result we've got that the expected *optimal* utility is 12.79926 which is better than $P1$'s 12.52149. We'd like to compare these results to the policy of *doing nothing*-the policy, according to which the investor puts all his money in the risk-free bond and simply waits. In this case the utility is given as $\frac{100^{0.2}}{0.2} = 12.559$. So we see that $P1$ is even worse than *doing nothing*!

Note that the small difference is due to the small interval length and to small value of α. By increasing all the values we can obtain the bigger differences. We don't provide here all the results, because it takes extremely long time to obtain some pretty pictures like this. For example in the above case the number N of iterations is about 170000.

In the second simulation the $P2$ policy is taken to be linear, it means that the boundaries are linear (the coefficients are taken to be the same). The parameters are

1. initial wealth $x = 1000000$,
2. $\varepsilon = 0.01$ and $r = 100$, so the real transaction cost is $k = r\varepsilon^4 = 0.000001$
3. the initial time is $t_0 = 0$ and the terminal $T = 10$, the time step dt is taken to be $dt = 0.001$, so the number of steps is 10000.
4. the process parameters are $\mu = 0.4$, $\sigma = 1$, $\alpha = 0.2$.

Figure 5.2: Comparing two policies (P2)



As a result we've got that the expected *optimal* utility is 92.77528 which is better than $P2$'s 86.76351. Comparing these two policies with the *doing nothing* policy value, which equals now $\frac{1000000^{0.2}}{0.2} = 79.244$ we see that *optimal* increase equals $92.7 - 79.2 = 13.5$ and the increase of $P2$ equals $86.7 - 79.2 = 7.5$ so it's almost double. The number of iterations in this simulation was about 200000.

6 Summary

In this work we have studied the problem of portfolio optimization when fixed transaction costs are present. In particular, we have looked at two problems- maximizing power utility (or CRRA utility) and the Goal Problem. The way of solving is via *impulse control* method but the fact that even using this method it's almost impossible to find an explicit solution it's desirable to find the way of approximating it. Here we use the approximation method, presented in [29], [30] and further in [13] [14].

6.1 Future research

The problems we've considered, have rather "friendly" strategies in the no - transaction costs case, and their value functions are known. All that makes us able to approximate the optimal policy. The approximation, though, looks nice but still there are lots of questions to ask, for instance the stationarity of the optimal strategy for the power utility is only due to the approximation and is still not obvious whether it could be achieved in the general case. The possible way to improve our approximations is maybe to go on and enlarge the power of the critical ε up to 6, for example. This will clearly add some complicated partial differential equations, and the solution will not be so explicit as now.

Also it's desirable to find the value function w or Q - the problem that we have avoided so far thanks to the fact that it was possible to do some simplifications and to derive the optimal strategy in a rather "simple" way.

Also we have to admit that solving the problem for the case when transaction is small makes it not so good in practice. For instance when the horizon is short- the improvement won't be so significant. So we have to make horizon longer (look at simulations) in order to see the difference.

All this leaves us still in search for the real solution - not approximated, even via numerical methods. For instance some recent works, like [6] with the numerical method of solving impulse control problems via iterated optimal stopping look rather promising, although extremely time-consuming. Also the paper [25] talks about new way of looking at these problems.

Impulse control itself looks rather interesting and promising for formulating and solving many other stochastic control problems.

7 Appendix - determining the approximation rule of the function Q

An important aspect of this work was to write the approximating value function in a manner that is complex enough to provide a solution to the QVI. The process of obtaining the form of the function Q consisted of trying simpler dependencies in powers of ϵ, with increasing complexity, until arriving the form of (3.6) which is used in this work. In this appendix we describe the failures to solve the QVI with simpler forms.

7.1 First order approximation

First it was natural to assume that the Y- term dependence appears in the ε - term, i.e. to assume the following form of the value function

$$Q(t,x,y) = \frac{1}{\alpha}\Big(x + \varepsilon G(t,x,y)\Big)^{\alpha} e^{\lambda(T-t)},$$

but it is not hard to get a contradiction here.

Consider the following derivative

$$Q_{yy} = \Big(\varepsilon G_{yy}(x+\varepsilon G)^{\alpha-1} + \varepsilon^2 (G_y)^2(\alpha-1)(x+\varepsilon G)^{\alpha-2}\Big)e^{\lambda(T-t)}.$$

Q_{yy} appears in equation (3.3) in only one term:

$$\frac{(u-1)^2}{\varepsilon^2} Q_{yy}.$$

Since this term is multiplied by ε, it follows (after the cancellation of the exponential term) that $\frac{1}{\varepsilon}G_{yy}$ must vanish, implying that $G_{yy} = 0$ and hence $G = Ay + B$ for some A and B which may depend on x and t but not on y. Further one can use the optimality of transaction condition (3.5) in order to get

$$G_y(t,x,Y^+) = 0 \text{ for all } x > k$$

and as a result obtain a contradiction, since it follows that A from the expression in G is equal to zero, and hence G does not depend on y.

7.2 Second order approximation

We now add the Y-dependency in the ε^2 term, that is assume the following form of the value function

$$Q(t,x,y) = \frac{1}{\alpha}\Big(x + \varepsilon H(t,x,y) + \varepsilon^2 G(t,x,y)\Big)^{\alpha} e^{\lambda(T-t)}.$$

Once again, the main problem was with the second derivative in y

$$Q_{yy} = \Big((\varepsilon H_{yy} + \varepsilon^2 G_{yy})(x + \varepsilon H + \varepsilon^2 G)^{\alpha-1} + ...\Big)e^{\lambda(T-t)}$$

which again appears in the main equation (3.3) in a single term:

$$\frac{(u-1)^2}{\varepsilon^2} Q_{yy}.$$

First of all, looking at the first order approximation, one immediately obtains that $H_{yy} = 0$ and hence H does not depend on y.

Consequently, when trying to make the $O(1)$ term vanish, we obtain the equation

$$\frac{1}{2}\sigma^2(ux)^2(\alpha-1) + \mu u x^2 - \frac{\lambda}{\alpha}x^2 + (u-1)^2 x G_{yy} = 0,$$

and using the fact that

$$\frac{1}{2}\sigma^2 u^2(\alpha-1) + \mu u - \frac{\lambda}{\alpha} = 0,$$

for

$$u = \frac{\mu}{\sigma^2(1-\alpha)} \qquad \text{and} \qquad \lambda = \frac{\mu^2\alpha}{2\sigma^2(1-\alpha)}$$

it follows that $G_{yy} = 0$ and we have arrived to the same contradiction as before.

7.3 Third order approximation

One has to work a little bit harder in order to show the contradiction in this case, where the function has the following structure

$$Q(t,x,y) = \frac{1}{\alpha}\Big(x + \varepsilon H(t,x,y) + \varepsilon^2 F(t,x,y) + \varepsilon^3 G(t,x,y)\Big)^{\alpha} e^{\lambda(T-t)}.$$

The only thing that can be obtained in a cheap way is that again the "problematic" Q_{yy}-term immediately gives us $H_{yy} = 0$. Then, from the $O(1)$ equation, (as in the second term approximation procedure above), it follows that $F_{yy} = 0$. All this means that neither H nor F depend on y. Taking this into consideration, we can write down the derivatives

$$Q_t = -\frac{\lambda}{\alpha}\Big(x + \varepsilon H + \varepsilon^2 F + \varepsilon^3 G\Big)^{\alpha} e^{\lambda(T-t)} +$$

$$+\Big(x + \varepsilon H + \varepsilon^2 F + \varepsilon^3 G\Big)^{\alpha-1}\Big(\varepsilon H_t + \varepsilon^2 F_t + \varepsilon^3 G_t\Big)e^{\lambda(T-t)},$$

$$Q_x = \Big(x + \varepsilon H + \varepsilon^2 F + \varepsilon^3 G\Big)^{\alpha-1}\Big(1 + \varepsilon H_x + \varepsilon^2 F_x + \varepsilon^3 G_x\Big)e^{\lambda(T-t)},$$

$$Q_y = \varepsilon^3 G_y\Big(x + \varepsilon H + \varepsilon^2 F + \varepsilon^3 G\Big)^{\alpha-1} e^{\lambda(T-t)},$$

$$Q_{xx} = (\alpha-1)\Big(x + \varepsilon H + \varepsilon^2 F + \varepsilon^3 G\Big)^{\alpha-2}\Big(1 + \varepsilon H_x + \varepsilon^2 F_x + \varepsilon^3 G_x\Big)^2 e^{\lambda(T-t)} +$$

$$+\Big(x + \varepsilon H + \varepsilon^2 F + \varepsilon^3 G\Big)^{\alpha-1}\Big(\varepsilon H_{xx} + \varepsilon^2 F_{xx} + \varepsilon^3 G_{xx}\Big)e^{\lambda(T-t)},$$

$$Q_{xy} = (\alpha-1)\Big(x + \varepsilon H + \varepsilon^2 F + \varepsilon^3 G\Big)^{\alpha-2}\varepsilon^3 G_y\Big(1 + \varepsilon H_x + \varepsilon^2 F_x + \varepsilon^3 G_x\Big)e^{\lambda(T-t)} +$$

$$+\Big(x + \varepsilon H + \varepsilon^2 F + \varepsilon^3 G\Big)^{\alpha-1}\varepsilon^3 G_{xy} e^{\lambda(T-t)},$$

$$Q_{yy} = \varepsilon^3 G_{yy}\Big(x + \varepsilon H + \varepsilon^2 F + \varepsilon^3 G\Big)^{\alpha-1} e^{\lambda(T-t)} +$$

$$+(\alpha-1)\Big(x + \varepsilon H + \varepsilon^2 F + \varepsilon^3 G\Big)^{\alpha-2}\varepsilon^6 (G_y)^2 e^{\lambda(T-t)}.$$

Now we have to put all these derivatives into equation (3.2).

Divide all the terms by $\Big(x+\varepsilon H+\varepsilon^2 F+\varepsilon^3 G\Big)^{\alpha-2} e^{\lambda(T-t)}$ and look at $O(\varepsilon)$ **term**:

$$\frac{1}{2}\sigma^2 u^2 x^2\Big((\alpha-1)2H_x + xH_{xx} + (u-1)^2 xG_{yy}\Big) - \sigma^2 uxy(\alpha-1)$$

$$+\mu ux\Big(H + xH_x\Big) - 2\frac{\lambda}{\mu}xH + xH_t - \mu xy = 0.$$

Once again we can see that for

$$u = \frac{\mu}{\sigma^2(1-\alpha)} \qquad \text{and} \qquad \lambda = \frac{\mu^2\alpha}{2\sigma^2(1-\alpha)}$$

always holds

$$\mu + \sigma^2 u(\alpha - 1) = 0, \quad \text{and} \quad \frac{1}{2}\sigma^2 u^2(\alpha - 1) + \mu u - \frac{\lambda}{\alpha} = 0$$

After these cancellations we get following differential equation for G:

$$\frac{1}{2}\sigma^2 u^2 x^2 \Big(H_{xx} + (u-1)^2 G_{yy} \Big) + H_t = 0$$

which can be stated as

$$G_{yy} = N(H)$$

and it follows that

$$G = \frac{1}{2} N y^2 + C y + D,$$

with all the variables N, C and D depending only on t and x.

Now the optimality conditions $G_y(t, x, Y^+) = 0$ and $G_y(t, x, \hat{Y}^+) = 0$ give us

$$N(Y^+ - \hat{Y}^+) = 0.$$

One may not assume that $N = 0$, since it immediately implies a contradiction, so it follows that $Y^+ = \hat{Y}^+$. However, the smoothness condition (3.9) implies a contradiction since we get $r = 0$ while assuming the transaction cost to be $k = \varepsilon^3 r$.

Recall that we have assumed that the fixed transaction cost would be $r\varepsilon^n$ for some power n and $\varepsilon^{(n+1)}$ is negligible. All this makes it clear that the highest power of the y-term must be the same as a ε - power n of the transaction.

7.4 Fourth order approximation

Moving further we add y-dependency in the ε^4-term:

$$Q(t,x,y) = \frac{1}{\alpha}\Big(x+\varepsilon H_1(t,x,y)+\varepsilon^2 H_2(t,x,y)+\varepsilon^3 H_3(t,x,y)+\varepsilon^4 G(t,x,y)\Big)^{\alpha} e^{\lambda(T-t)}.$$

After all the work in the previous subsections, almost in the same way, one gets that neither one of H_1, H_2 nor H_3 depends on y.

One can simplify the form of the value function even more by recalling that the transaction cost now equals to $k = r\varepsilon^4$ which makes the value function to be symmetric in $-\varepsilon$ and ε. So it follows that all odd powers of epsilon (not depending on y of course) should be zero and hence H_1, H_3 may be ignored.

So we've shown that the first term where the y-dependency can enter the equation without causing any contradiction is in the ε^4-term.

So finally we have arrived to the form

$$Q(t,x,y) = \frac{1}{\alpha}\Big(x + \varepsilon^2 H(t,x) + \varepsilon^4 G(t,x,y) + ...\Big)^{\alpha} e^{\lambda(T-t)}.$$

References

[1] Atkinson, C. and Wilmott, P. (1995). Portfolio management with transaction costs: an asymptotic analysis. *Mathematical Finance 5*(4) , 357-367.

[2] Benssounssan, A. and Lions, J.L. (1984). Impulse control and quasi-variational inequalities. Paris: Gauthier-Villars .

[3] Bielecki, T. and Pliska. S.R. (2000). Risk sensitive management with transaction costs. *Finance and Stochastics 4,* 1-33.

[4] Borodin, A. and Salminen, P. Handbook of Brownian Motion: Facts and Formulae. Boston, MA: Birkhauser, 1996.

[5] Buckley, I. and Korn, R. (1998). Optimal index tracking under transaction costs and impulse control. *International journal of theoretical and applied Finance 1*(3), 315-330.

[6] Chancelier, J.-P. Oksendal, B., Sulem, A. (2000). Combined stochastic control and optimal stopping, with applications to portfolio optimization under fixed transaction costs. *Pre-print*.

[7] Constantinides, G.M. (1976). Stochastic cash management with fixed and proportional transaction costs. *Management Science* (22), 1320-1331.

[8] Davis, M.H. and Norman, A.R. (1990). Portfolio selection with transaction costs. *Mathematics of Operation Research 15* (4), 676-713.

[9] Eastham, J.E. and Hastings, K.J. (1988). Optimal impulse control of portfolios. *Mathematics of Operation Research 13* , 588-605.

[10] Heath, D. (1993), A continuous-time version of Kulldorff's result, *Unpublished manuscript*.

[11] Karatzas, I. Lectures on the mathematical finance. *CRM monograph series, ISSN 1065-8599; Vol.8* , 1997.

[12] Kannan, D. Lakshmikantham, V. Handbook of stochastic analysis and applications. *Marcel Dekker, Inc.* 2002.

[13] Korn, R. (1998). Portfolio optimization with strictly positive transaction costs and impulse control. *Finance and Stochastics 2,* 85-114.

[14] Korn, R. (1999). Some applications of impulse controls in mathematical finance. *Mathematical Methods of Operations Research.* (50), 493-518.

[15] Kulldorff, M. (1993). Optimal control of favorable games with a time limit. *SIAM Journal on Control and Optimization 31*(1), 52-69.

[16] Leland, H.E. (2000). Optimal portfolio implementation with transaction costs and capital gains taxes. *Working paper, U.C.Berkeley.*

[17] Liu, H.(2002) Optimal consumption and investment with transaction costs and multiple stocks. *Working paper. Olin School of Business, Washington University of St. Louis.*

[18] Magill, M.J.P. and Constantinides, G.M. (1976). Portfolio selection with transaction costs. *Journal Econ. Theory 13* 245-263.

[19] Merton R.C. (1969). Lifetime portfolio selection under uncertainty : the continuous-time model. *Rev. Econ. Statist. 51*, 247-257.

[20] Merton, R.C. (1971). Optimum consumption and portfolio rules in a continuous-time model. *Journal Econ. Theory 3* , 373-413.

[21] Morton, A.J. and Pliska. S.R. (1995). Optimal portfolio management with transaction costs. *Mathematical Finance 5*(4) , 337-356.

[22] Oksendal, B., Sulem, A. (2000). Introduction to Impulse Control Theory and Application to Economics. Lecture Notes, *INRIA / University of Oslo.*

[23] Oksendal, B. Stochastic Differential Equations. *5th edition. Springer-Verlag* 2000.

[24] Oksendal, B., Sulem, A. (2002). Optimal Consumption and Portfolio with Both Fixed and Proportional Transaction Costs. *SIAM Journal on Control and Optimization 40*(6), 1765-1790.

[25] Oksendal, A. (2000), A semigroup approach to impulse control problems. *Working paper.*

[26] Pliska. S.R. and Suzuki, K. (2002) Optimal tracking for asset allocation with fixed and proportional transaction costs. (pre-print)

[27] Shreve,S.E. and Soner, H.M. (1994) Optimal investment and consumption with transaction costs. *Annals of Applied Probability 4*(3), 609-692.

[28] Taksar, M., Klass, M.J. and Assaf, D. (1988). A diffusion model for optimal portfolio selection in the presence of the brokerage fees. *Mathematics of Operation Research 13*(2), 277-294.

[29] Whalley, A.E. and Wilmott, P. (1997). An asymptotic analysis of an optimal hedging model for optimal pricing with transaction costs,. *Mathematical Finance 7* (3), 307-324.

[30] Whalley, A.E. and Wilmott, P. (1999). Optimal hedging of options with small but arbitrary transaction cost structure. *Euro.Jnl. of Applied Matthematics 10*, 117-139.