Math 54 Summer 2008 Section 005 Midterm 2

Your Name:______________________________

Explain all your work. Answers without proper justification will not get credit.

Problem 1 *(10pts) Find the equation of the line that best fits the given points* $(-1,1),(0,2),(1,5)$ *in the least-squares sense.*

We are looking for a and b such that $y=ax+b$ best fits these points. I.e. we would like a & b to satisfy

$(-1,1)$ $1=-a+b$

$(0,2)$ $2=0a+b$

$(1,5)$ $5=a+b.$

which is equivalent to

$$\begin{pmatrix}-1 & 1\\ 0 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}1\\ 2\\ 5\end{pmatrix}$$

The least squares solution ~~will be~~ is given by

$$\begin{pmatrix}a\\ b\end{pmatrix}=\left(\begin{pmatrix}-1 & 0 & 1\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}-1 & 1\\ 0 & 1\\ 1 & 1\end{pmatrix}\right)^{-1}\begin{pmatrix}-1 & 0 & 1\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}1\\ 2\\ 5\end{pmatrix}=$$

$$=\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}^{-1}\begin{pmatrix}4\\ 8\end{pmatrix}=\begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{3}\end{pmatrix}\begin{pmatrix}4\\ 8\end{pmatrix}=\begin{pmatrix}2\\ \frac{8}{3}\end{pmatrix}.$$

So the line is $y=2x+\frac{8}{3}$

Problem 2 *(10pts) Let* $A = \begin{pmatrix} 4 & 3 \\ -5 & -4 \end{pmatrix}$. *Find* A^{25}.

If A is diagonalizable, $A = PDP^{-1}$, then

$A^{25} = PD^{25}P^{-1}$, so let's try to diagonalize A.

To do that, we need to find ~~a~~ a basis of $\mathbb{R}^2$

consisting of ~~eigenvalues~~ eigenvectors of A.

To find the eigenvalues, solve $\det(A - \lambda I) = 0$ for λ.

$$\det\left(\begin{pmatrix} 4 & 3 \\ -5 & -4 \end{pmatrix} - \lambda I\right) = \det\begin{pmatrix} 4-\lambda & 3 \\ -5 & -4-\lambda \end{pmatrix} = (4-\lambda)(-4-\lambda) - 3(-5) =$$

$$= \lambda^2 - 16 + 15 = \lambda^2 - 1$$ so the eigenvalues are ± 1.

$\lambda = 1$. Finding $\mathrm{Null}(A - I)$ will give the corresponding eigenvectors.

$A - I = \begin{pmatrix} 3 & 3 \\ -5 & -5 \end{pmatrix}$.

$$\begin{pmatrix} 3 & 3 & 0 \\ -5 & -5 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ The eigenvectors are $\begin{pmatrix} -t \\ t \end{pmatrix}$.

$\lambda = -1$ $\mathrm{Null}(A + I)$.

$A + I = \begin{pmatrix} 5 & 3 \\ -5 & -3 \end{pmatrix}$

$$\begin{pmatrix} 5 & 3 & 0 \\ -5 & -3 & 0 \end{pmatrix} \sim \begin{pmatrix} 5 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ The eigenvalues are $\begin{pmatrix} -\frac{3}{5}t \\ t \end{pmatrix}$.

So a basis of eigenvectors is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 5 \end{pmatrix}$.

$$A = \begin{pmatrix} -1 & -3 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & -3 \\ 1 & 5 \end{pmatrix}^{-1}. \quad A^{25} = \begin{pmatrix} -1 & -3 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{25}\begin{pmatrix} -1 & -3 \\ 1 & 5 \end{pmatrix} =$$

$$= \begin{pmatrix} -1 & -3 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & -3 \\ 1 & 5 \end{pmatrix} = A = \begin{pmatrix} 4 & 3 \\ -5 & -4 \end{pmatrix}. \quad A^{25} = \begin{pmatrix} 4 & 3 \\ -5 & -4 \end{pmatrix}.$$

Problem 3 *(10pts) Let $T : P_2 \to P_3$ be the linear transformation*

$$(a + bx + cx^2) \to (2a + b + c + (c - b)x + bx^3).$$

Find the matrix that gives this linear transformation if we choose the bases

$$\mathfrak{B} = \{1 - x, x - x^2, x^2\} \text{ for } P_2 \text{ and } \mathfrak{C} = \{1, 1 + x, x + x^2, x^2 + x^3\} \text{ for } P_3.$$

The matrix is given by $M = \left([T(\bar{b}_1)]_e \mid [T(\bar{b}_2)]_e \mid [T(\bar{b}_3)]_e\right)$.

$$T(\bar{b}_1) = T(1-x) = 1 + x - x^3$$
$$T(\bar{b}_2) = T(x - x^2) = -2x + x^3$$
$$T(\bar{b}_3) = T(x^2) = 1 + x.$$

Find $[T(\bar{b}_1)]_C$

$$1 + x - x^3 = c_1 \cdot 1 + c_2(1+x) + c_3(x+x^2) + c_4(x^2+x^3)$$
$$1 + x - x^3 = c_1 + c_2 + (c_2+c_3)x + (c_3+c_4)x^2 + c_4x^3$$

$$\left.\begin{matrix} c_4 = -1, \\ c_3 + c_4 = 0 \\ c_2 + c_3 = 1 \\ c_1 + c_2 = 1 \end{matrix}\right\} \Rightarrow \begin{matrix} c_4 = -1 \\ c_3 = 1 \\ c_2 = 0 \\ c_1 = \phi. \end{matrix} \quad \text{, so } [T(\bar{b}_1)]_C = \begin{pmatrix} \phi \\ 0 \\ 1 \\ -1 \end{pmatrix}.$$

Find. $[T(\bar{b}_2)]_C$

$$-2x + x^3 = c_1 \cdot 1 + c_2(1+x) + c_3(x+x^2) + c_4(x^2+x^3)$$
$$-2x + x^3 = (c_1+c_2) + (c_2+c_3)x + (c_3+c_4)x^2 + c_4x^3.$$

$$\left.\begin{matrix} c_4 = 1 \\ c_3 + c_4 = 0 \\ c_2 + c_3 = -2 \\ c_1 + c_2 = 0 \end{matrix}\right\} \Rightarrow \begin{matrix} c_4 = 1 \\ c_3 = -1 \\ c_2 = -1 \\ c_1 = 1 \end{matrix} \quad [T(\bar{b}_2)]_C = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

Find $[T(\bar{b}_3)]_C$.

$$1 + x = c_1 \cdot 1 + c_2(1+x) + c_3(x+x^2) + c_4(x^2+x^3)$$
$$c_1 = 0,\ c_2 = 1,\ c_3 = 0,\ c_4 = 0. \quad [T(\bar{b}_3)]_C = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

So The matrix of T w.r.t. $\mathcal{B}, \mathcal{C}$ is $M = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix}.$

Problem 4 *(10pts) Apply the Gram-Schmidt process to the vectors*

$$f_1(x) = 1, f_2(x) = x, f_3(x) = x^2,$$

if

$$f \cdot g = \int_{-1}^{1} f(x)g(x)dx.$$

$g_1(x) = 1$

$g_2(x) = f_2(x) - \dfrac{f_2(x) \cdot g_1(x)}{g_1(x) \cdot g_1(x)} g_1(x).$

$f_2(x) \cdot g_1(x) = \int_{-1}^{1} x\,dx = \frac{x^2}{2}\Big|_{-1}^{1} = 0.$, so $g_2(x) = x - 0 = x.$

$g_3(x) = f_3(x) - \dfrac{f_3(x)\, g_1(x)}{g_1(x)\, g_1(x)} g_1(x) - \dfrac{f_3(x)\, g_2(x)}{g_2(x)\, g_2(x)} g_2(x).$

$f_3(x) \cdot g_1(x) = \cancel{\int_{-1} x^4 dx = \frac{x^5}{5}\Big|_{-1}^{1} = \frac{2}{5}} = \int_{-1}^{1} x^2 dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3}$

$g_1(x) \cdot g_1(x) = \cancel{\int_{-1} x^2 dx = \frac{x^3}{3}\Big|_{-1}^{1} = 2} = \int_{-1}^{1} 1^2 dx = x\Big|_{-1}^{1} = 2$

$f_3(x) \cdot g_2(x) = \int_{-1}^{1} x^3 dx = x^4\Big|_{-1}^{1} = 0.$

So $g_3(x) = x^2 - \dfrac{\frac{2}{3}}{2} 1 + 0x = x^2 - \frac{1}{3}$

Let's normalize these.

$h_1(x) = \dfrac{g_1(x)}{\|g_1(x)\|}$. $\|g_1(x)\| = \sqrt{\int_{-1}^{1} 1\,dx} = \sqrt{x\Big|_{-1}^{1}} = \sqrt{2}$. . $h_1(x) = \frac{1}{\sqrt{2}}$

$h_2(x) = \dfrac{g_2(x)}{\|g_2(x)\|}$ $\|g_2(x)\| = \sqrt{\int_{-1}^{1} x^2 dx} = \sqrt{\frac{x^3}{3}\Big|_{-1}^{1}} = \sqrt{\frac{2}{3}}$. $h_2(x) = x\sqrt{\frac{3}{2}}$

$h_3(x) = \dfrac{g_3(x)}{\|g_3(x)\|}$ $\|g_3(x)\| = \sqrt{\int_{-1}^{1} (x^2 - \frac{1}{3})^2 dx} = \sqrt{\int_{-1}^{1} x^4 - \frac{2x^2}{3} + \frac{1}{9}\, dx} = \sqrt{\frac{x^5}{5} - \frac{2x^3}{9} + \frac{x}{9}\Big|_{-1}^{1}} =$

$= \sqrt{\frac{2}{5} - \frac{4}{9} + \frac{2}{9}} = \sqrt{\frac{8}{45}} =$ $h_3(x) = (x^2 - \frac{1}{3}) \sqrt{\frac{45}{8}}$

Problem 5 *(10pts) For all parts except (b) and (e) answer TRUE or FALSE, or YES or NO. No need to justify your answer. You will get 1 point for each correct answer and -0.25 points for each incorrect answer, with maximum 10 and minimum 0 total points for the problem.*

NO (a) *Let* $v_1, v_2, v_3, v_4, v_5 \in \mathbb{R}^5$. *If* $\{v_1, v_2, v_3\}$ *is a linearly independent set and* $\{v_4, v_5\}$ *is a linearly independent set, is* $\{v_1, v_2, v_3, v_4, v_5\}$ *a linearly independent set?*

(b) *Circle the ones that ar equal to* $u \cdot v$. *(here* u *and* v *are* $n \times 1$ *column vectors and* 1×1 *matrices are identified with numbers) :*

$$u^T v \qquad uv^T \qquad v^T u \qquad vu^T$$

NO (c) *Are there vectors* $u, v \in \mathbb{R}^5$ *such that* $||u|| = 3, ||v|| = 2, ||u + v|| = 6$?

NO (d) *Let* $v_1, v_2, v_3, v_4, v_5 \in \mathbb{R}^5$. *If* $\{v_1, v_2\}$ *is an orthonormal set,* $\{v_3, v_4, v_5\}$ *is an orthonormal set, is* $\{v_1, v_2, v_3, v_4, v_5\}$ *an orthonormal set?*

(e) *If* $B = \{v_1, v_2, v_3, v_4, v_5\}$ *is an orthogonal basis for* $\mathbb{R}^5$, *and*

$$v = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 + a_5 v_5,$$

express a_2 *in terms of inner products.*

$$a_2 = \frac{V \cdot V_2}{V_2 \cdot V_2}$$

Yes (f) *Can a* 2×2 *matrix have more than 3 eigenvectors?*

Yes (g) *Is* $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix}$ *diagonalizable?*

NO (h) *If* u *and* v *are eigenvectors of a symmetric matrix, must they be orthogonal?*

Yes (i) *If* $\{v_1, v_2\}$ *is an orthonormal set of eigenvectors of a symmetric matrix* A *with eigenvalue 1, and* $\{v_3, v_4, v_5\}$ *is an orthonormal set of eigenvectors of* A *with eigenvalue -1, is* $\{v_1, v_2, v_3, v_4, v_5\}$ *an orthonormal set?*

Yes (j) *If* A *is a* 5×5 *matrix with characteristic polynomial* $-(\lambda - 2)^3(\lambda - 1)^2$ *then* $\dim Null A = 0$.

Problem 6 *(10pts) Find the eigenvalues and eigenvectors of the linear transformation $T : C^\infty[0,1] \to C^\infty[0,1]$ given by $T(y) = y'$. ($C^\infty[0,1]$ denotes the space of infinitely differentiable functions with domain $[0,1]$).*

If $y = y(x)$ is an eigenvector with eigenvalue λ, then

$T(y) = \lambda y$, so $y' = \lambda y$.

$\cancel{\frac{dy}{dx} = \lambda y}$

The functions whose derivative ~~is a const~~ doesn't change except for multiplying by a constant are $e^{\lambda x}$. (~~Another~~ One way to see this is

$$y' = \lambda y$$
$$\frac{dy}{dx} = \lambda y$$
$$\frac{dy}{y} = \lambda dx$$
$$\int \frac{dy}{y} = \int \lambda dx.$$
$$\ln|y| = \lambda x + c_1$$
$$y = c_2 e^{\lambda x}$$
)

So any real number λ is an eigenvalue, and the eigenvectors for this eigenvalue are $ce^{\lambda x}$ where c is any non-zero constant.