

PRACTICE MIDTERM 2 SOLUTIONS

Note: This practice midterm doesn't have the format of the midterm. It is simply a collection of problems you should know how to solve. Some of them could have been on the midterm, as I was considering them, but for one reason or another, didn't include.

Problem 1. Does the following define an inner product on M_{22} ?

$$(A, B) = \text{trace}(A)\text{trace}(B).$$

(The trace of a square matrix is the sum of its diagonal entries)

Solution 1. To see whether this defines an inner product, we need to check if it satisfies the properties an inner product should have. In this problem all the properties hold, except the one that says

$$(u, u) = 0 \Leftrightarrow u = 0.$$

For example if we take

$$u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then we have $(u, u) = \text{tr}(u)\text{tr}(u) = 0 \cdot 0 = 0$ but $u \neq 0$. So $(A, B) = \text{tr}(A)\text{tr}(B)$ doesn't define an inner product on M_{22} .

If you needed to check that for example the property

$$(A + B, C) = (A, C) + (B, C)$$

is satisfied, this is how you would do it.

$$\begin{aligned} (A + B, C) &= \text{tr}(A + B)\text{tr}(C) = ((A + B)_{11} + \dots + (A + B)_{nn})(C_{11} + \dots + C_{nn}) = \\ &= (A_{11} + B_{11} + \dots + A_{nn} + B_{nn})(C_{11} + \dots + C_{nn}) = \\ &= ((A_{11} + \dots + A_{nn}) + (B_{11} + \dots + B_{nn}))(C_{11} + \dots + C_{nn}) = \\ &= (A_{11} + \dots + A_{nn})(C_{11} + \dots + C_{nn}) + (B_{11} + \dots + B_{nn})(C_{11} + \dots + C_{nn}) = \\ &= \text{tr}(A)\text{tr}(C) + \text{tr}(B)\text{tr}(C) = (A, C) + (B, C). \end{aligned}$$

Problem 2. Let $T : M_{22} \rightarrow \mathbb{R}_2$ be the linear transformation given by

$$A \mapsto (\text{trace}(A), \text{sum of the elements of } A).$$

Find the matrix that gives this linear transformation if we choose the standard bases for M_{22} and $\mathbb{R}^2$.

Solution 2. The matrix of a linear transformation with respect to bases $\mathfrak{B} = \{b_1, \dots, b_p\}$ for the domain and $\mathfrak{C}$ for the codomain is given by

$$M = \begin{pmatrix} [T(b_1)]_{\mathfrak{C}} & \dots & [T(b_p)]_{\mathfrak{C}} \end{pmatrix}.$$

$$\text{In our case } [T(b_1)]_{\mathfrak{C}} = \left[T \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) \right]_{\mathfrak{C}} = (1, 1)_{\mathfrak{C}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$[T(b_2)]_{\mathfrak{C}} = \left[T \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) \right]_{\mathfrak{C}} = (0, 1)_{\mathfrak{C}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$[T(b_3)]_{\mathfrak{E}} = \left[T \left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right) \right]_{\mathfrak{E}} = (0, 1)_{\mathfrak{E}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$[T(b_4)]_{\mathfrak{E}} = \left[T \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) \right]_{\mathfrak{E}} = (1, 1)_{\mathfrak{E}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

so the matrix M is

$$M = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Problem 3. Find the equation of the line that best fits the given points $(-2, -2), (-1, 0), (0, -2), (1, 0)$ in the least-squares sense.

Solution 3. Suppose $y = ax + b$ is the equation of the line that best fits these points. Then we have

$$\begin{aligned} -2 &= -2a + b \\ 0 &= -a + b \\ -2 &= 0a + b \\ 0 &= a + b \end{aligned}$$

. If there was a line fitting these points, a, b would be solutions of

$$\begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ -2 \\ 0 \end{bmatrix}.$$

The least squares solution of this equation is given by

$$\begin{aligned} \begin{bmatrix} a \\ b \end{bmatrix} &= \left(\begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}^T \begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}^T \begin{bmatrix} -2 \\ 0 \\ -2 \\ 0 \end{bmatrix} = \\ &= \left(\begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} \right)^{-1} \begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}^T \begin{bmatrix} -2 \\ 0 \\ -2 \\ 0 \end{bmatrix} = \frac{1}{6 \cdot 4 - (-2)(-2)} \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 4 \\ -4 \end{bmatrix} = \\ &= \frac{1}{20} \begin{bmatrix} 8 \\ -16 \end{bmatrix} = \begin{bmatrix} 2/5 \\ -4/5 \end{bmatrix}. \end{aligned}$$

So, the line is $y = 2/5x - 4/5$.

Problem 4. Find the projection of the vector $(1, 2, 3)$ and the zx plane in $\mathbb{R}^3$. What is the distance from $(1, 2, 3)$ to the zx plane?

Solution 4. Since $\left\{ u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is an orthonormal basis of the zx plane, the projection of $v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ on the zx plane is given by

$$\text{proj } v = (v \cdot u_1)u_1 + (v \cdot u_2)u_2 = 1u_1 + 3u_2 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}.$$

The distance from v to the zx plane is the same as the length of $v - \text{proj } v$ so it is

$$\left\| \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \right\| = \sqrt{0^2 + 2^2 + 0^2} = 2.$$

Problem 5. Show that $S = \{u_1, u_2, u_3\}$ is an orthonormal basis for $\mathbb{R}^3$, where $u_1 = (2/3, 2/3, 1/3)$, $u_2 = (\sqrt{2}/2, -\sqrt{2}/2, 0)$, $u_3 = (\sqrt{2}/6, \sqrt{2}/6, -2\sqrt{2}/3)$. Using the fact that S is an orthonormal basis, find the coordinates of $v = (1, 1, 1)$ in this basis.

Solution 5. Hint. Check that $(u_1, u_2) = (u_1, u_3) = (u_2, u_3) = 0$ and that $(u_1, u_1) = (u_2, u_2) = (u_3, u_3)$. These imply that the coordinates of v in this basis are $(v, u_1), (v, u_2), (v, u_3)$. Calculate these.

Problem 6. Apply the Gram-Schmidt process to the vectors

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

The inner product is $A \cdot B = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}$.

Solution 6. Call these vectors x_1, x_2, x_3 . Take $v_1 = x_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. By the Gram-Schmidt process

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 = x_2 - \frac{0}{6} v_1 = x_2 = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}.$$

$$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} - \frac{2}{4} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \frac{-1}{6} \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1/6 & 2/3 \\ -1/3 & -1/2 \end{pmatrix}.$$

Now, we can normalize these vectors to get

$$u_1 = \frac{v_1}{\|v_1\|} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix},$$

$$u_2 = \frac{v_2}{\|v_2\|} = \begin{pmatrix} -2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{6} & 0 \end{pmatrix},$$

$$u_3 = \frac{v_3}{\|v_3\|} = \begin{pmatrix} 1/(6\sqrt{5/6}) & 2/(3\sqrt{5/6}) \\ -1/(3\sqrt{5/6}) & -1/(2\sqrt{5/6}) \end{pmatrix}.$$

Problem 7. Let $A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$. Find A^{25} .

Solution 7. Hint. Find the eigenvalues of A and a basis of eigenvectors. Using these diagonalize A . If $A = PDP^{-1}$ then $A^{25} = PD^{25}P^{-1}$.

Problem 8. Find an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors of $A = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & -2 \\ 2 & -2 & 0 \end{pmatrix}$.

Solution 8. First, let's find the eigenvalues. To do that, we can find the characteristic polynomial of A and find its roots. $\det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 2 & 2 \\ 2 & -\lambda & -2 \\ 2 & -2 & -\lambda \end{pmatrix} =$

$-\lambda^3 + 12\lambda - 16 = -(\lambda + 4)(\lambda - 2)^2$, so the eigenvalues are $-4, 2$. Now, let's find the eigenvectors. For $\lambda = -4$ we need to find the nullspace of $A + 4I$.

$$\begin{pmatrix} 4 & 2 & 2 & 0 \\ 2 & 4 & -2 & 0 \\ 2 & -2 & 4 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & -1 & 0 \\ 1 & -1 & 2 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & -1 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & -1 & 2 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -3 & 3 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

You can calculate the solutions to be $\begin{pmatrix} -t \\ t \\ t \end{pmatrix}$. So, a basis for this eigenspace is

given by $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$. For $\lambda = 2$ we get

$$\begin{pmatrix} -2 & 2 & 2 & 0 \\ 2 & -2 & -2 & 0 \\ 2 & -2 & -2 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so the eigenvectors will be $\begin{pmatrix} s+t \\ s \\ t \end{pmatrix}$, and a basis for the 2-eigenspace is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

So, $\mathfrak{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$ is a basis of $\mathbb{R}^3$ consisting of eigenvectors

of A . Now, let's find an orthonormal basis of eigenvectors. Note, that if you just apply the Gram-Schmidt process to $\mathfrak{B}$ the resulting vectors although orthonormal, might not be eigenvectors anymore. Instead, let's apply the process to each eigenspace separately. For the eigenspace for eigenvalue -4 , an orthonormal basis

is $u_1 = \begin{pmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$. For the eigenvalue 2 we get

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix}.$$

Normalizing these we get $w_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 1/(2\sqrt{3/2}) \\ -1/(2\sqrt{3/2}) \\ 1/\sqrt{3/2} \end{pmatrix}$. So, we have that

$\{u_1\}$ is an orthonormal basis for the -4 -eigenspace, and $\{w_1, w_2\}$ is an orthonormal basis for the 2-eigenspace. Since A is symmetric, u_1 is automatically orthogonal to w_1 and w_2 (this can also be checked directly), so $\{u_1, w_1, w_2\}$ will be an orthonormal basis of $\mathbb{R}$ consisting of eigenvectors of A .

Problem 9. Find the eigenvalues and eigenvectors of the linear transformation $T: M_{22} \rightarrow M_{22}$ given by $A \mapsto A^T$.

Solution 9. A matrix A would be an eigenvector, if $T(A) = \lambda A$ for some λ . I.e. $A^T = \lambda A$. By taking transpose of both side, we see that $A = (\lambda A)^T = \lambda A^T = \lambda^2 A$. Since A is assumed to be an eigenvector, it is not 0, so $\lambda^2 = 1$, i.e. only ± 1 could be eigenvalues. Let's try to find the eigenvectors. When $\lambda = 1$, A is an eigenvector, if $A^T = \lambda A = A$. Writing $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we can see that saying $A^T = A$ is exactly the same as saying $b = c$, so all the matrices $\begin{pmatrix} a & b \\ b & d \end{pmatrix}$ are eigenvectors with eigenvalue 1. Similarly, for $\lambda = -1$, we see that all the matrices $\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$ satisfy $A^T = -A$, so they are eigenvectors with eigenvalue -1 .

Problem 10. Let A be a 4×4 matrix, u_1, u_2 linearly independent vectors in $\mathbb{R}^3$. Suppose $Au_1 = Au_2 = 0$. Which of the following are true?

- (1) A can not be invertible.
- (2) $\dim(CS(A)) = 2$.
- (3) $rk(A) \leq 2$.
- (4) A has at most 3 distinct eigenvalues.
- (5) A is not diagonalizable.

Solution 10. .

- (1) True. If A were invertible, $Ax = 0$ would have only the solution $x = 0$ but here we know that u_1, u_2 are solutions. They are not 0 because they form a linearly independent set.
- (2) False $u_1, u_2 \in \text{Null}(A)$, so $\dim \text{Null}(A) \geq 2$. By the rank nullity theorem $\dim(CS(A)) \leq 4 - 2 = 2$. So it doesn't have to be 2, it can be less.
- (3) True $rk(A) = \dim(CS(A))$, so the above calculation shows this is true.
- (4) True. The sum of the dimensions of the eigenspaces can be at most 4. We see that u_1, u_2 are eigenvectors with eigenvalue 0, so we know that the dimension of the 0 eigenspace is at least 2. If there were more than 2 other eigenvalues, the sum of dimensions of eigenvectors would be more than 4, so there are at most 2 other eigenvalues.
- (5) False. For example take A to be the zero matrix.

Problem 11. The characteristic polynomial of A is $-\lambda^3 - \lambda$. Which of the following is true?

- (1) A is definitely diagonalizable?
- (2) A is definitely not diagonalizable?
- (3) A may or may not be diagonalizable?

Solution 11. The answer is (2), since if A were diagonalizable, it would have 3 eigenvalues (counting with multiplicities) but from the characteristic polynomial we can see that that's not the case.

Problem 12. (True or False) If $x \cdot y = 0$ for all $y \in \mathbb{R}^3$ then x must be $\bar{0}$.

Solution 12. True. $x \cdot y = 0$ for all $y \in \mathbb{R}^3$, so in particular this is true when $y = x$. I.e. $x \cdot x = 0$, which implies $x = 0$.

Problem 13. Let $T : C[-1, 1] \rightarrow C[-1, 1]$ be the linear transformation given by $(T(f))(x) = f(-x) \forall x \in [-1, 1]$. Find the eigenvectors and eigenvalues of T .

Solution 13. A function f would be an eigenvector, if $T(f) = \lambda f$ for some λ . I.e. $f(-x) = \lambda f(x), \forall x \in [-1, 1]$. Using this we see that if f is an eigenvector with eigenvalue λ , then $f(x) = \lambda f(-x) = \lambda^2 f(x)$. Since $f(x)$ is assumed to be an eigenvector, it is not 0, so $\lambda^2 = 1$, i.e. only ± 1 could be eigenvalues. Let's try to find the eigenvectors. When $\lambda = 1$, $f(x)$ is an eigenvector, if $f(-x) = \lambda f(x) = f(x)$. This is exactly the same as saying f is an even function. Similarly, for $\lambda = -1$, we see that exactly all the odd functions satisfy $f(-x) = \lambda f(x) = -f(x)$, so they are the eigenvectors with eigenvalue -1 .