Practice midterm 1
Solutions

Problem 1:

Let $A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$. Then V is

$$V = \left\{ \text{all} \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \;\middle|\; \begin{matrix} 3a+3b+4c=0 \\ 3d+3e+4f=0 \end{matrix} \right\}$$

From here $a = -b - \frac{4}{3}c$, $d = -e - \frac{4}{3}f$,

So Another way to write V is.

$$V = \left\{ \begin{pmatrix} -b-\frac{4}{3}c & b & c \\ -e-\frac{4}{3}f & e & f \end{pmatrix} \;\middle|\; \text{any } b,c,e,f \right\}$$

From here

$$V = \left\{ b\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c\begin{pmatrix} -\frac{4}{3} & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + e\begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix} + f\begin{pmatrix} 0 & 0 & 0 \\ -\frac{4}{3} & 0 & 1 \end{pmatrix} \;\middle|\; \text{any } b,c,e,f \right\}$$

The matrices

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -\frac{4}{3} & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ -\frac{4}{3} & 0 & 1 \end{pmatrix}$$

are linearly independent and V is their span, therefor they form a basis for V and $\dim V = 4$.

Problem 2: V will be a vector space if $0 \in V$, and V is closed under addition and scalar multiplication, since V is a subset of $\mathbb{P}_3$.

Let's check if V is closed under addition.

Suppose $p, q \in V$. Then

$$p(3) = 2p(-3) + 3$$
$$q(3) = 2q(-3) + 3.$$

From these we see

$$(p+q)(3) = p(3) + q(3) = 2p(-3) + 3 + 2q(-3) + 3 =$$
$$= 2(p+q)(-3) + 6.$$

So $(p+q)(3) \neq 2(p+q)(-3) + 3$.

which implies V is not closed under addition $\Rightarrow$ V is not a vector space.

Problem 3:

a) False. Even though the rows of A can be linearly independent, they don't have to be.

Counterexample: $A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

b) True. ~~A is a square matrix.~~

A has three rows and they are linearly dependent $\Rightarrow$ $\dim \operatorname{Row} A < 3$. But $\dim \operatorname{Row} A = \dim \operatorname{Col} A \Rightarrow \dim \operatorname{Col} A < 3 \Rightarrow$ since A has 3 columns, they must be linearly dependent.

Problem 4:

To find the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$ we ~~can~~ can calculate $C^{-1}B$ where

$$C = \begin{pmatrix} 2 & 6 \\ -1 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} -6 & 2 \\ -1 & 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|cc} 2 & 6 & -6 & 2 \\ -1 & -2 & -1 & 0 \end{array}\right) \overset{\text{swap}}{\sim} \left(\begin{array}{cc|cc} -1 & -2 & -1 & 0 \\ 2 & 6 & -6 & 2 \end{array}\right) \underset{r2 \to r2 + 2\cdot r1}{\sim}$$

$$\sim \left(\begin{array}{cc|cc} -1 & -2 & -1 & 0 \\ 0 & 2 & -8 & 2 \end{array}\right) \underset{r2 \to \frac{1}{2} r2}{\overset{r1 \to -r1}{\sim}} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 8 & -2 \end{array}\right) \sim$$

$$\underset{r1 \to r1 + (-2) r2}{\sim} \left(\begin{array}{cc|cc} 1 & 0 & -15 & 4 \\ 0 & 1 & 8 & -2 \end{array}\right).$$

$C^{-1}B = \begin{pmatrix} -15 & 4 \\ 8 & -2 \end{pmatrix}$. This is the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$.

Problem 5:

First, let's show that $Av_1, \ldots, Av_n$ are linearly independent.

Suppose $c_1 Av_1 + \cdots + c_n Av_n = 0$.

This implies $A(c_1 v_1 + \cdots + c_n v_n) = 0$.

Since A is invertible, the only solution to $Ax = 0$ is $x = 0$. This implies $c_1 v_1 + \cdots + c_n v_n = 0$.

But $v_1, \ldots, v_n$ form a basis for $\mathbb{R}^n \Rightarrow$ they are linearly independent $\Rightarrow c_1 = c_2 = \cdots = c_n = 0$.

This establishes that $Av_1, \ldots, Av_n$ are linearly independent.

Now, we have ~~that~~ n linearly independent vectors in $\mathbb{R}^n$. Since $\dim \mathbb{R}^n = n$, any set of ~~any~~ n linearly independent vectors will form a basis, so $Av_1, \ldots, Av_n$ form a basis of $\mathbb{R}^n$.