Math 54 Summer 2006

Group-work problems. Lec 11.

1. Is there a vector space that has
   - no vectors
   - 1 vector
   - 2 vectors

2. Let L and M be subspaces of a vector space V. Are the following sets also subspaces of V?
   - $L \cup M = \{$ all vectors belonging to L or M or both $\}$.
   - $L \cap M = \{$ all vectors belonging to both L and $M\}$.
   - $L + M = \{$ all vectors w that can be written as a sum of a vector x in L and a vector y in $M\}$.

   Let $L = \{(x, y, x)|x + y + z = 0\}$ and let $M = \{(x, y, z)|x = y = z\}$. Both are subspaces of $\mathbb{R}^3$. Give geometric descriptions of L and M, then describe $L \cup M, L \cap M, L + M$.

3. Degine the differential operator $D = \frac{d}{dx} - 1$ by the rule: $D(f) = (\frac{d}{dx-1})f = \frac{df}{dx} - 1$. To make this rigorous, one may think of D as a function whose domain and range is the vector space $C^\infty(\mathbb{R})$ of infinitely differentiable funcions defined on $\mathbb{R}$. As with matrices, one can define the null space of D by $NS(D) = \{f \in C^\infty(\mathbb{R}) : D(f) = 0\}$.
   - Show that $NS(D)$ is a subspace of $C^\infty(\mathbb{R})$.
   - Find $NS(D)$. (Hint: Solve a differential equation).

4. Suppose that $S = \{v_1, \dots, v_k\}$ and $T = \{w_1, \dots, w_l\}$ are subsets of a vector space V. Furthermore, S and T are each linearly independent sets, and $Span(S) \cap Span(T) = \{0\}$. Show that $S \cup T$ is a linearly independent set.

5. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
   - Show that $A^2 - (a + d)A + (ad - bc)I = 0$. Thus $\{I, A, A^2\}$ is a linearly dependent set in M_{22}.
   - Deduce that A^n is in $Span\{I, A\}$ for all non-negative integers n. Show that if A is invertible, then A^n is in $Span\{I, A\}$ even if n is a negative integer.

6. Let V be the vector space of all polynomials in P_3 satisfying $p(1) = 0$. Give a basis for V and find $dimV$.

7. Let $V = C(\mathbb{R})$ be the space of continuous functions $f : \mathbb{R} \to \mathbb{R}$.

- Show that $C(\mathbb{R})$ is infinite dimensional.
- Let W be the subspace spanned by $sin^2x, cos^2x, sin2x, cos2x$. What is $dimW$?

8. Let $V = P_5$ and let W be the set of polynomials in V which are even functions. Is W a subspace of V? If so, find a basis for W.

9. Let W be a subspace of $\mathbb{R}^n$ and let $\{w_1, \ldots, w_k\}$ be a linearly independent subset of W.
   - Suppose that $\{w_1, \ldots, w_k, w\}$ is linearly dependent for each w in W. Show that $\{w_1, \ldots, w_k\}$ is a basis for W.
   - Suppose moreover that $\{w_1, \ldots, w_k, v\}$ is linearly dependent for each v in $\mathbb{R}^n$. Show that $k = n$.

10. Suppose that A is an invertible $n \times n$ matrix. What is $rk(A)$? What is $dimNS(A)$?

11. Show that if A is not square, then either the rows of A or the columns of A are linearly dependent.

12. Let A be the matrix formed by taking n vectors from $\mathbb{R}^m$ as its columns. In each of the following situations, what relationship must hold between m and n? Choose from: $m \leq n, n \leq m$ and $m = n$.
    - The n vectors are linearly independent.
    - The n vectors span $\mathbb{R}^m$.
    - The n vectors form a basis for $\mathbb{R}^m$.
    - A has rank m.
    - A is invertible.

13. Find the rank of the following matrices (it may depend on t):

$$A = \begin{bmatrix} 1 & 1 & t \\ 1 & t & 1 \\ t & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} t & -1 & 2 \\ t & t & 1 \\ t & t & t \end{bmatrix}.$$

14. - Suppose that u is a non-zero $m \times 1$ matrix, and v is a non-zero $1 \times n$ matrix. Show that $A = uv$ is an $m \times n$ matrix of rank 1.
    - Show that the converse is true also. That is, suppose that A is an $m \times n$ matrix of rank 1. Show that there is a column matrix u and a row matrix v so that $A = uv$.