Math 21-325 - Probability

Homework Assignment 12
Due Dec 7

1. (a) Let $N, X_1, X_2, \dots$ be independent $\mathbb{Z}_{>0}$ valued random variables, with $X_1, X_2, \dots$ i.i.d. Show that the generating function of the random sum

   $$S(\omega) := \sum_{k=1}^{N(\omega)} X_k(\omega)$$

   is

   $$G_S(s) = G_N(G_X(s)),$$

   where $G_X = G_{X_i}$ for all i.

   (b) Let N be a Poisson random variable with rate λ. Let S be the number of successes in N Bernoulli trials, where the probability of success is $p \in (0, 1)$. What is the distribution of S?

2. Let $Y_1, Y_2, \dots$ be independent, identically distributed variables, each of which can take any value in $0, 1, 2, \dots, 9$ with equal probability $1/10$. Let $X_n = \sum_{i=1}^{n} Y_i 10^{-i}$. Show by the use of characteristic functions that X_n converges in distribution to the uniform distribution on $[0, 1]$. Independently, show that $X_n \xrightarrow{n\to\infty} Y$ almost surely, for some random variable Y. Now, deduce that Y is uniformly distributed in $[0, 1]$.

3. (Chebyshev's inequality)

   Let $c > 0$. Prove that $P(|X| \geq c) \leq \frac{1}{c^p} E(|X|^p)$ for all $p \geq 1$. Note, that we have seen the case $p = 2$ in class.

4. We have seen three notions of convergence of random variables in class. Here is one more. A sequence of random variables $X_1, X_2, \dots$ converges to X in L^p if

   $$\lim_{k=\infty} E(|X_k - X|^p) = 0.$$

   When $p = 2$ this is also often called mean-square convergence. When $p = 1$ it is called convergence in mean.

   Prove that if $X_n \to X$ in L^p for some $p \geq 1$, then $X_n \to X$ in probability.

5. (Law of large numbers) Let $X_1, X_2, d \dots$ be random variables with $E(X_k) = \mu \in \mathbb{R}$ and variance $Var(X_k) = \sigma^2 \in \mathbb{R}$. Prove, that if the random variables are uncorrelated (note, that this is weaker than independent), then

$$\frac{1}{n} \sum_{k=1}^{n} X_k \xrightarrow{n \to \infty} \mu \text{ in } L^2.$$

   In class we prove convergence in probability under weaker assumptions.

6. Let $(B_t)_{t \in \mathbb{R}_+}$ be Brownian Motion. What is the correlation coeffitient and joint distribution of B_t and B_s, where $0 \leq t < s$? What is $P(B_t > 0, B_s > 0)$?

7. For what values of a and b is $aW_1 + bW_2$ a Wiener process (this is another name for Brownian motion), where W_1 and W_2 are independent Wiener processes?

8. Show, that the standard Brownian motion has finite quadratic variation, i.e. that

$$\sum_{j=0}^{n-1} (B_{(j+1)t/n} - B_{jt/n})^2 \xrightarrow{n \to \infty} t \quad \text{in } L^2.$$