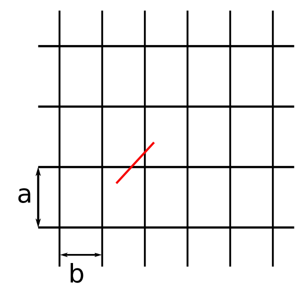
Math 21-325 - Probability

Homework Assignment 11
Due Nov 30

1. Find the probability that $x^2 + 2ax + b$ has complex roots if the coefficients a and b are independent random variables with density
$$f(x) = 1_{x>0}\alpha e^{-\alpha x}.$$

2. Two grids of parallel lines are superimposed: the first grid contains lines distance a apart, the second contains lines distance b apart which are perpendicular to those of the first set. A needle of length $r < \min(a, b)$ is dropped at random. Find the probability that it intersects a line.



Hint: Identify the needle by its midpoint and angle, which are independent and uniformly random.

3. Construct an example of two random variables X and Y for which $\mathbb{E}(Y) = \infty$ but such that $\mathbb{E}(Y|X) < \infty$ almost surely, i.e. $P(\mathbb{E}(Y|X) < \infty) = 1$.

4. Find the conditional density function and expectation of Y given X when they have joint density function
$$f(x, y) = xe^{-x(y+1)}1_{x\geq 0}1_{y\geq 0}.$$

5. Let X and Y have the bivariate normal density function $N(0, 1, \rho)$. Show that X and $Z = (Y - \rho X)/\sqrt{1-\rho^2}$ are independent standard normal random variables, and deduce that
$$P(X > 0, Y > 0) = \frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\rho.$$

6. The random variable X has the gamma distribution with parameters $\lambda, t > 0$, denoted by $\Gamma(\lambda, t)$, if it has density

$$f(x) = \frac{1}{\Gamma(x)} \lambda^t x^{t-1} e^{-\lambda x}, \qquad x \geq 0,$$

where $\Gamma(x)$ is the gamma function defined by

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx.$$

Let $X_1, X_2, \ldots, X_n$ be exponential random variables (see Hwk 10) with parameter λ. Show by induction that $S = X_1 + \ldots X_n$ has the $\Gamma(\lambda, n)$ distribution.

7. A random variable with distribution $\Gamma\left(\frac{1}{2}, \frac{n}{2}\right)$ is said to have the chi-squared distribution $\chi^2(n)$ with n degrees of freedom. Let $X_1, X_2, \ldots, X_n$ be independent standard normal random variables.

   (a) Show that X_1^2 is $\chi^2(1)$.

   (b) Show that $X_1^2 + X_2^2$ is $\chi^2(2)$ by expressing its distribution function as an integral and changing to polar coordinates.

   (c) More generally, show that $X_1^2 + X_2^2 + \ldots X_n^2$ is $\chi^2(n)$.

   This is saying that if the coordinates of a vector $\bar{x} \in \mathbb{R}^n$ are standard normals, then the square of the length of $\bar{x}$ is $\chi^2(n)$.