Final Exam
Math 371 Fall 2010

Instructions: You are allowed to work on this exam a maximum of 5 hours. You may not consult any people, notes, books or other resources for the exam. You can submit the exam to my office or scan and email me your work. If you choose this option you have to make certain that a printout of your submission will be easily legible.

The last day to submit the exam is December 15, 2010. Attach a signed copy of this frontpage to your submission.

Write Your Name as it appears on Owl Space:

________________________ ________________________
Last Name First Name

Honor Pledge: On my honor, I have neither given nor received any unauthorized aid on this exam.

Sign:

Problem	Score
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1. (5pts) Define what a maximal torus is.

2. (5pts) How are rotations of $\mathbb{R}^3$ be represented in terms of unit quaternions?

3. (5pts) Describe the map $Sp(1) \times Sp(1) \to SO(4)$ geometrically.

4. (5pts) Is the set of real numbers $\mathbb{R}$ under addition isomorphic to a matrix Lie group? Justify your answer.

5. (5pts) Define the group $U(2,3)$.

6. (5pts) Let G and H be path connected matrix Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. If $\mathfrak{g}$ is isomorphic to $\mathfrak{h}$ as a Lie algebra, must G and H be isomorphic as Lie groups? If yes, prove it, if no, give a counter example.

7. (5pts) Prove that for any matrix Lie group G there exists $\delta > 0$ such that every $X \in N_\delta(I)$ has an nth root, i.e. there is $Z \in G$ such that $Z^n = X$.

8. (5pts) Prove that for any matrix Lie group G there exists $\delta > 0$ such that for every $X, Y \in N_\delta(I)$ if $e^X e^Y = e^Y e^X$ then $XY = YX$.

9. (10pts) Give geometric interpretations for the groups $SO(n,\mathbb{R})$, $SL(n,\mathbb{R})$, $U(n)$, $Sp(n)$, $Sp(2n,\mathbb{C})$. For example the group $SO(n,\mathbb{R})$ is the group of orientation preserving orthogonal linear transformations of the space $\mathbb{R}^n$, where orthogonality means preservation of the inner product $(u,v) = u^T v$.

10. (10pts) Let H be the matrix Lie group which consists of matrices

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}.$$

Describe the Lie algebra $\mathfrak{h}$ of H. Is $\mathfrak{h}$ a simple Lie algebra? If yes, prove it, if no, give a non-trivial ideal.

11. (10pts) Give an example of a connected matrix Lie group with a nondiscrete normal subgroup H such that $T_I(H) = \{0\}$.

12. (15pts) Prove that the complexificatoin of $sp(n)$ is $sp(2n,\mathbb{C})$.

13. (15pts) Let $Sp_J(2n,\mathbb{C})$ be the symplectic Lie group corresponding to the symplectic form $(u,v)_s^J = u^T J_n v$ and $Sp_\Omega(2n,\mathbb{C})$ be the symplectic Lie group corresponding to the symplectic form $(u,v)_s^\Omega = u^T \Omega_n v$, where J_n is block diagonal consisting of n blocks of J. Here

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

and I_n is the identity matrix of size n.

Prove that $Sp_\Omega(2n,\mathbb{C})$ and $Sp_J(2n,\mathbb{C})$ are isomorphic. (Hint: Find a matrix B such that $B^T J B = \Omega$ and use B to define an explicit isomorphism.)