(1) Prove that the set of functions of the form \( f(x) = a_0 + a_1x + a_2x^2 \) for \( a_0, a_1, a_2 \in \mathbb{Q} \) is countable.

**Solution:** Let \( P = \{ f \in \mathbb{R}^\mathbb{R} \mid \exists a_0, a_1, a_2 \in \mathbb{Z}, f(x) = a_0 + a_1x + a_2x^2 \} \). We show that \( P \) is countable by defining a surjection from \((\mathbb{Q} \times \mathbb{Q}) \times \mathbb{Q}\) to \( P \), i.e. \( f((a, b), c) = a + bx + cx^2 \).

**Lemma:** \((\mathbb{Q} \times \mathbb{Q}) \times \mathbb{Q}\) is countable.

**Proof:** We showed in lecture that \( \mathbb{Q} \) is countable, and we know that the cartesian product of two countable sets is countable, thus we know that \( \mathbb{Q} \times \mathbb{Q} \) is countable, so for the same reason, \((\mathbb{Q} \times \mathbb{Q}) \times \mathbb{Q}\) is countable.

(2) Let \( n \geq 4 \). How many ways can you assign \( n \) grad students to 3 offices such that each office has assigned at least 2 students?

*Note that the problem should have specified \( n \geq 4 \), otherwise there are too many cases to consider in a reasonable amount of time.*

**Solution:** Let \( U \) be the set of all ways to assign the \( n \) grad students to the 3 offices. Let \( A_1 \) be the set of all ways to assign the \( n \) grad students to the 3 offices, such that office number 1 is assigned at most 1 student. Define \( A_2 \) and \( A_3 \) in the same way.

Then \( U - (A_1 \cup A_2 \cup A_3) \) is the set of all ways to assign the grad students such that each office has assigned at least 2 students.

Then \( |U - (A_1 \cup A_2 \cup A_3)| = |U| - |A_1| - |A_2| - |A_3| + |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| - |A_1 \cap A_2 \cap A_3| \), by Inclusion-Exclusion.

\( |U| = 3^n \)

\( |A_1| = 2^n + n2^{n-1} \) since there are \( 2^n \) ways to assign the grad students such that office 1 gets no one assigned, and \( n2^{n-1} \) ways such that office 1 gets exactly one grad student assigned.

\( |A_1 \cap A_2| = 1 + n + n(n-1) \) since there is one way to assign the grad students such that offices 1 and 2 get 0 students, \( n \) ways such that office 1 gets 1 student and office 2 gets 0 students, \( n \) ways such that office 2 gets 1 student and office 1 gets 0 student, and \( n(n-1) \) ways such that both offices 1 and 2 get 1 student each.
\[ |A_1 \cap A_2 \cap A_3| = 0 \] since there is no way to assign student such that all three offices have at most student, because \( n \geq 4 \).

Since \( |A_1| = |A_2| = |A_3| \), and \( |A_1 \cap A_2| = |A_1 \cap A_3| = |A_2 \cap A_3| \), we have

\[ |U - (A_1 \cup A_2 \cup A_3)| = 3^n - 3(2^n + n2^{n-1}) + 3(1 + 2n + n(n - 1)) \]