Disclaimer: I have designed this review sheet to aid you in studying for the midterm. However, the problems in this document do **not** emcompass all of the kinds of questions you may be asked in the midterm. In addition to doing these problems, you should also review problems from your homeworks, discussion sessions, quizzes, lecture, and the textbook.

We will discuss problems of your choice on Thursday.

1. Logic and Proof Techniques

(1) Show that $(2n+1)^2 - 1$ is a multiple of 8 for every natural number n.
(2) Show that $4x^4 + 2y^4 = z^4$ does not have any integer solutions, except for $x, y, z = 0$.
(3) Show that if x and y are non-negative real numbers such that $x \leq y$, then $\sqrt{x} \leq \sqrt{y}$.
(4) Let $p(x)$ be a logical formula. Prove that the following are all logically equivalent:
   (a) $(\exists x, p(x)) \wedge (\forall y, \forall z, (y \wedge z) \to y = z)$
   (b) $\exists x, (p(x) \wedge \forall y, (p(y) \to y = x))$
(5) For each $a \in \mathbb{R}$, there is a unique $x \in \mathbb{R}$ such that $x^2 + 2ax + a^2 = 0$.
(6) Let $A = \{1, 2\}$, $B = \{1, 2\}$, $C = \{\emptyset, 1\}$. Compute $A \cup (B \times C)$.
(7) Write the contrapositive of the following proposition: "If x and y are both odd integers, then there is no integer z such that $x^2 + y^2 = z^2$".
(8) Write the proposition $(p \leftrightarrow q) \to (q \leftrightarrow r)$ in CNF.

2. Sets

(1) Let A, B, and X be sets such that $A \subseteq X$ and $B \subseteq X$. Either prove or disprove:

$$X \setminus (A \Delta B) = (X \setminus A) \Delta (X \setminus B).$$

(2) For each $x \in \mathbb{R}$, let $P_x = \{x^n \mid n \in \mathbb{N}.\}$. Compute the following:
   (a) $\bigcup_{x \in [0,1]} P_x$
   (b) $\bigcap_{x \in [0,1]} P_x$
   (c) $\bigcap_{m \in [N]} P_{2^m}$ where $N \in \mathbb{N}$
   (d) $\bigcap_{n \in \mathbb{N}} P_{2^n}$
(3) Let A, B, and X be sets such that $A \subseteq X$ and $B \subseteq X$. Show that $(A \cup B) \cap (X \setminus A) = B \setminus A$.
(4) Which one of the following statements is the true one? Prove your answer.
   (a) $\mathcal{P}(\mathbb{R}) \supsetneq \bigcup_{r \in \mathbb{R}_{\geq 0}} \mathcal{P}([-r, r])$
   (b) $\mathcal{P}(\mathbb{R}) \subsetneq \bigcup_{n \in \mathbb{R}_{\geq 0}} \mathcal{P}([-r, r])$
   (c) $\mathcal{P}(\mathbb{R}) = \bigcup_{r \in \mathbb{R}_{\geq 0}} \mathcal{P}([-r, r])$

3. Induction

(1) Show that for every $n \geq 1$, $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$.
(2) Let $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for every $n \geq 2$. Show that for every $n \geq 1$, $f_{n-1}f_{n+1} = f_n^2 + (-1)^n$
(3) Show that for every $n \geq 1$, $\sum_{k=n}^{2n-1} 2k + 1 = 3n^2$
(4) Show that $n^3 < 3^{n-1}$ for every $n \geq 1$.

(5) Let $a_0 = 0$, $a_1 = 1$, and $a_n = 5a_{n-1} - 6a_{n-2}$. Prove $a_n = 3^n - 2^n$ for every $n \geq 0$.

(6) Compute the set $\{3k + 5\ell \mid k, \ell \in \mathbb{N}\}$. Prove your answer.

4. Functions

(1) Let $f : \mathbb{Z} \to \mathbb{Z}$ be the function

$$f(x) = \begin{cases} 0 & x \leq 0 \\ x^2 & 0 \leq x \leq 10 \\ -1 & x > 10 \end{cases}.$$

What is $f(\mathbb{Z})$?

(2) Let $f(x) = 2^x$. In which of the following cases is f well-defined? injective? surjective? bijective?

(a) $f : \mathbb{R} \to \mathbb{R}$

(b) $f : \mathbb{R} \to \mathbb{R}^{>0}$

(c) $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{>0}$

(d) $f : \mathbb{R} \to \mathbb{Z}$

(3) For each of the following, decide if it is the graph of a function $f : X \to Y$ where $X = [-1, 1]$, $Y = [-1, 1]$.

(a) $G_1 = \{(x, y) \in X \times Y \mid x^2 + y^2 = 1\}$

(b) $G_2 = \{(x, x^3) \mid x \in X\}$

(c) $G_3 = \{(x^3, x) \mid x \in X\}$