Disclaimer: I have designed this review sheet to aid you in studying for the final. However, the problems in this document do **not** emcompass all of the kinds of questions you may be asked in the exam. In addition to doing these problems, you should also review problems from your homeworks, discussion sessions, quizzes, lecture, and the textbook.

We will discuss problems of your choice on Thursday.

1. Functions

(1) For each of the following functions, decide if it is well-defined? a surjection? injection? bijection?
   (a) $f : \mathcal{P}(\mathbb{R}) \times \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ where
$$f(A, B) = A \cup B;$$
   (b) $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ where
$$f(a, b) = 2^a 3^b;$$
   (c) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{N}$ where
$$f(x, y) = (q, r) \text{ where } x = qy + r \text{ and } 0 \leq r < y.$$
(2) Exhibit an explicit bijection between each of the following pairs of sets:
   (a) $A = \mathbb{Z}$, $B = \mathbb{N}$;
   (b) $A = \mathbb{R} \setminus \{0\}$, $B = \mathbb{R}$;
   (c) $A = \mathcal{P}([5])$, $B = [32]$.

2. Cardinality

(1) For each of the following sets, decide if it is finite, countably-infinite, or uncountable.
   (a) $\mathcal{P}(\mathbb{N} \times \mathbb{N})$;
   (b) $\bigcup_{n\in\mathbb{N}} \mathcal{P}([n])$;
   (c) $\mathcal{P}(\bigcup_{n\in\mathbb{N}}[n])$;
   (d) $\{f : \mathbb{R} \to \mathbb{R} \mid f(0) = 0\}$;
   (e) $\bigcup_{n\in\mathbb{N}} \mathbb{N}^{[n]}$.

3. Pigeonhole Principle

(1) Show that every set of five points in the unit square has a pair of points of distance at most $\sqrt{2}/2$.
(2) Determine the maximum size of a subset of [99] that has no two elements differing by 3.
(3) Let S be a set of n integers. Show that there must be a subset of S whose sum is divisible by n.

4. Counting

(1) How many binary strings of length m have exactly k ones?
(2) How many ways can n pirates distribute k gold doubloons?
(3) How many non-negative integer solutions are there to
$$x_1 + x_2 + ... + x_k \leq n?$$

(4) Prove the following by counting in two ways:

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.$$

5. Number Theory

(1) Compute all integer solutions to the equation $18x + 30y = 28$.
(2) Compute the multiplicative inverse of 3 mod 11.
(3) Compute the order of 5 mod 11.
(4) Show that if p is a positive prime number and $x^2 \equiv 1 \mod p$, then either $x \equiv 1 \mod p$ or $x \equiv -1 \mod p$.
(5) Prove that the only integer solution to the equation $3n^2 = 40m^2$ is $(0, 0)$.
(6) Compute the last digit of 2^{21127}.
(7) Describe all integers x that satisfy the congruence

$$3x + 4 \equiv x \mod 6.$$

(8) Prove that $n(n^2 - 1)(n^2 + 1)$ is divisible by 5 for any $n \in \mathbb{N}$.
(9) Show that for every $n \in \mathbb{N}$, there exists an integer a such that $a, a+1, a+2, ..., a+n$ are all composite (not prime).
(10) Find all integer solutions to the system of congruences:

$$\begin{cases} x \equiv 3 & \mod 7 \\ x \equiv 5 & \mod 10 \\ 2x \equiv 3 & \mod 9 \end{cases}$$