1. Important Problems

1.1. **Cardinality.**

(1) Recall that if X and Y are sets, then X^Y is the set of all functions from Y to X.
Show that $\mathbb{N}^{\mathbb{N}}$ is uncountable (i.e. not countable).
(2) Recall that $[0] = \emptyset$ and $[n] = \{1, 2, ..., n\}$ for $n \geq 1$. Show that $\bigcup_{n \in \mathbb{N}} [n]$ is countable.
(3) Show that the set of polynomials with one variable, with integer coefficients, is countable. In other words, show that the set of functions $f : \mathbb{R} \to \mathbb{R}$ of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$$

for some $n \in \mathbb{N}$ and $a_0, a_1, ..., a_n \in \mathbb{Z}$.
Hint: First, show that for a given natural number n, the set of polynomials of degree at most n is countable. Do this by induction on n.
(4) Quickly find a subset of [13] not listed below, with at least 9 elements.
- $\{7, 12, 13, 2, 3, 8, 1, 9\}$
- $\{6, 10, 7, 13, 1, 9, 2, 12\}$
- $\{12, 9, 13, 4, 2, 8, 1, 6, 10\}$
- $\{1, 7, 8, 11, 4, 9, 5, 12\}$
- $\{3, 1, 9, 5, 4, 2, 12, 7\}$
- $\{8, 10, 12, 7, 4, 2, 9, 13, 6, 1, 5, 3\}$
- $\{13, 10, 12, 9, 3, 8, 4, 11, 7, 5, 2, 1\}$
- $\{5, 3, 12, 2, 13, 4, 9, 11, 10, 1, 8, 6\}$
- $\{7, 8, 10, 1, 3, 6, 12, 2, 13\}$
- $\{6, 13, 3, 5, 4, 9, 2, 11\}$
- $\{12, 3, 1, 11, 4, 9, 13, 7, 8, 6, 10\}$
- $\{12, 11, 1, 5, 7, 8, 9, 13, 10, 6, 4, 2\}$
- $\{5, 9, 10, 2, 3, 12, 7, 6, 4, 8, 1, 11\}$

1.2. **Pigeonhole Principle and Inclusion-Exclusion.**

(1) Consider the n-by-n grid of dots at positions $\{(i, j) \mid 1 \leq i \leq n, 1 \leq j \leq n\}$ in the plane. Each dot is black or white. How large does n have to be such that you can always find a rectangle whose corners all have the same corner? Show that your choice of n is best possible by giving an example of a coloring of the $(n-1)$-by-$(n-1)$ grid with no such rectangle.
(2) How many ways are there to assign 7 grad students to 3 offices? How many ways can you assign 7 grad students to 3 offices such that every office has at least one grad student?

2. Extra Problems

2.1. **Cardinality.**

(1) Show that if I is a countable set, and for each $i \in I$, A_i is a countable set, then $\bigcup_{i \in I} A_i$ is countable.

(2) Show that $\mathbb{Q}^{\mathbb{Q}}$ is uncountable.

(3) Show that for any set X, there is *never* a bijection between X and $\mathcal{P}(X)$.

(4) Show that the set of irrational real numbers is uncountable.

(5) Let $\mathbb{N}^1 = \mathbb{N}$ and for $n \geq 2$, let $\mathbb{N}^n = \mathbb{N} \times \mathbb{N}^{n-1}$. Show that for each $n \in \mathbb{N}$, $\mathbb{N}^n$ is countable. Show that furthermore, $\bigcup_{n \in \mathbb{N}} \mathbb{N}^n$ is countable.

2.2. **Pigeonhole Principle and Inclusion-Exclusion.**

(1) Show that in a class of 6 people, there must be a set of 3 people that they all know each other, or they all don't know each other.

(2) On a field 400 yards long, ten people each mark off football fields of length 100 yards. Prove there must be a point that belongs to at least 4 fields.

(3) Prove that every set of five points in the square of area 1 has two points separated by a distance of at most $\sqrt{2}/2$.

(4) Two integers p and q are relatively prime if the only positive divisor they both have is 1. How many natural numbers less than 252 are relatively prime to 252?

(5) Mary, Samantha, and John are distributing 100 dollars among themselves. How many ways can they do this such that Mary gets at least 5 dollars, and Samantha gets at most 5 dollars?