1. Important Problems

(1) Prove or disprove: For real numbers a and b, if $4ab < (a+b)^2$ then $a < b$

(2) Prove that for integers a, b, and c, if $a^2 + b^2 = c^2$ then either a or b is even.

(3) Show that for all integers a and b, that $a^2 - a - b^2 + b$ is even.

(4) Let A be some set of integers numbers. Write out symbolically the proposition that says that there are at least two different ways to represent the number 3 as the sum of elements of A.

If A contained the numbers 4 and -1, we consider $4+(-1)$ and $(-1)+4$ to be the same representation.

(5) Let $p(x)$ and $q(x)$ be variable propositions and S a set. Which of the following pairs of propositions are logically equivalent? If they are logically equivalent, prove it. Otherwise, give a counterexample.

(a) $\forall x \in S, (p(x) \wedge q(x))$ and $(\forall x \in S, p(x)) \wedge (\forall x \in S, q(x))$

(b) $\forall x \in S, (p(x) \vee q(x))$ and $(\forall x \in S, p(x)) \vee (\forall x \in S, q(x))$

(c) $\exists x \in S, (p(x) \wedge q(x))$ and $(\exists x \in S, p(x)) \wedge (\exists x \in S, q(x))$

(d) $\exists x \in S, (p(x) \vee q(x))$ and $(\exists x \in S, p(x)) \vee (\exists x \in S, q(x))$

(6) Write the contrapositive of the following statements:

(a) For integers x, y and z, if x divides both y and z, then x divides $y + z$.

(b) If x is a multiple of 3 and a multiple of 5, then x is a multiple of 15.

(c) If $x > 1$ then $x^2 > x$.

(7) In the following variable propositions, x ranges over the integers $\mathbb{Z}$. Let $p(x)$ be the proposition "$1 \leq x \leq 3$". Let $q(x)$ be the proposition "$\exists k \in \mathbb{Z}(x = 2k)$". Let $r(x)$ be the proposition "$x^2 = 4$". Let $s(x)$ be the proposition "$x = 1$".

For each of the following statements, write the logical negation in negation normal form. Then decide which claim (the original or the negation) is true.

(a) $\forall x \in \mathbb{Z}, ((s(x) \wedge r(x)) \to (s(x) \wedge q(x)))$

(b) $\exists x \in \mathbb{Z}, ((s(x) \wedge p(x) \wedge q(x)) \vee (\exists y \in \mathbb{Z}, (y \neq x \wedge r(x) \wedge r(y) \wedge p(x) \wedge p(y)))$

(c) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, s(y) \vee r(x)$

2. More Problems

(1) We define the connective $\oplus$ with the following truth table:

p	q	$p \oplus q$
T	T	F
F	T	T
T	F	T
F	F	F

This operation is called 'xor' or 'exclusive or'.

Express $x \oplus y$ using x and y and some combination of the connectives $\wedge$, $\vee$, $\neg$.

Show that $x \oplus y$ is logically equivalent to $y \oplus x$. Show that $(x \oplus y) \oplus z$ is logically equivalent to $x \oplus (y \oplus z)$. Finally, show that $(x \oplus y) \oplus x$ is logically equivalent to y.

(2) Determine whether the following are true or false. Prove your answer.

(a) For every x there is a y such that $xy = 0$.

(b) There exists an x such that for every y, $x + y = 0$
(c) For all n, either n is even or n is odd
(d) For all x there exists y such that $xy = 1$
(e) For all $x \neq 0$ there exists y such that $xy = 1$
(f) For all x there exists y such that $x + y = 0$
(g) Either for all n, n is even, or for all n, n is odd.

(3) Detertmine where the following are true or false. Prove your answer.
(a) If a and b are irrational, so is $a + b$
(b) If a and b are irrational, so is ab
(c) If r is rational and a is irrational, then $a + r$ is irrational
(d) if r is rational and a is irrational, then ar is rational
(e) if r is rational and a is irrational, then ar is irrational
(f) if r is rational and s is rational then r^s is rational
(g) if r is rational and s is rational and nonzero, then r/s is rational

(4) Show that assuming what you want to prove and then concluding something true is NOT a valid proof strategy, by showing that an appropriate logical statement is not a tautology.

(5) Let Q and P be propositional formulae. Define a propositional formula R such that P and Q are logically equivalent if and only if R is a tautology.

(6) Write the following English-language stories into propositional formulae, using the following propositions:
- r - It was raining
- w - I went outside
- j - I brought my jacket

(a) I went for a walk. If it was raining, I brought my jacket.
(b) I was going for a walk when it was raining, but I wasn't wearing my rain jacket.
(c) I forgot my rain jacket, but luckily I did not get rained on today.