This document will be used to keep track of our development of counting finite and infinite sets. I will update it as we go along.

1. Finite Sets

Recall that $[0] = \emptyset$ and $[n] = \{1, 2, ..., n\}$ for $n \geq 1$.

Theorem 1. *Let $m, n \in \mathbb{N}$. Then the following all hold:*

(1) If there is an injection $f : [m] \to [n]$ then $m \leq n$.
(2) If there is a surjection $f : [m] \to [n]$ then $m \geq n$.
(3) If there is a bijection $f : [m] \to [n]$, then $m = n$.

Proof. We showed (1) in lecture. You will do (2) in Assignment 5. And (3) follows immediately from (1) and (2). □

Definition 1. If X is a set and $n \in \mathbb{N}$, we say that X has cardinality n (denoted $|X| = n$) if there is a bijection $f : [n] \to X$. We say X is *finite* if $|X| = n$ for some $n \in \mathbb{N}$. We say X is infinite if it is not finite.

Theorem 2. *Let A be a set. Then the following are equivalent:*

(1) A is finite.
(2) There is an injection $f : A \to [n]$ for some $n \in \mathbb{N}$.
(3) There is a surjection $f : [n] \to A$ for some $n \in \mathbb{N}$.

Proof. It is enough to prove (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1). You can accept this theorem with out proof, though by the end of the week you should make sure you know how to prove this. We will cover a proof of a very similar theorem which will give you the main ideas. □

Theorem 3. *If X and Y are finite sets, and there exists a bijection $f : X \to Y$, then $|X| = |Y|$*

Proof. You will prove this in Assignment 5. □

Theorem 4. *Let X be a nonempty set such that $x \in X$ and $y \notin X$. Suppose that $|X| = n$ for some natural number n. Then $|X \cup \{y\}| = n + 1$ and $|X \setminus \{x\}| = n - 1$.*

Proof. You will prove this in Assignment 5. □

Theorem 5. *Every non-empty set of natural numbers has a least element.*

Proof. We proved this in lecture. □

Theorem 6. *Every non-empty finite set of natural numbers has a greatest element.*

Proof. The proof of this is very similar to the proof of the last theorem. This problem will be in the discussion session on Friday. □

Theorem 7. $\mathbb{N}$ *is infinite.*

Proof. We proved this in lecture. □

Theorem 8. *If X and Y are sets, such that $X \subseteq Y$, and X is infinite, then so is Y.*

Proof. You should make sure you can prove this! □

Theorem 9. *Let X and Y be sets. Then the following hold:*

(1) If $f : X \to Y$ is an injection and X is infinite, then $f[X]$ is infinite.
(2) If $f : X \to Y$ is a surjection and Y is infinite, then $f^{-1}[Y]$ is infinite.

Proof. Proved in lecture. □

2. Countability

Definition 2. A set X is *countably infinite* if there exists a bijection $f : \mathbb{N} \to X$. We say that X is *countable* if it is finite or countably infinite.

Theorem 10. $\mathbb{N}$ *and* $\mathbb{Z}$ *are countably infinite.*

Proof. Proved in class. □

Theorem 11. $\mathbb{N} \times \mathbb{N}$ *is countably infinite.*

Proof. Proved in class, using a result from your homework. □

Theorem 12. *Let X be a non-empty set. Then the following are equivalent:*

(1) X is countable.
(2) There exists a surjection $f : \mathbb{N} \to X$.
(3) There exists an injection $f : X \to \mathbb{N}$.

Proof. (Laborously) proved in class. □

Theorem 13. *Let X be a non-empty set. Then the following are equivalent:*

(1) X is countable.
(2) There exists a surjection $f : C \to X$ for some countable set C.
(3) There exists an injection $f : X \to C$ for some countable set C.

Theorem 14. *If A and B are countable sets, then so is $A \times B$.*

Proof. Proved in class. □

Theorem 15. $\mathbb{Q}$ *is countable.*

Proof. Proved in class. □

3. Uncountability

Theorem 16. $\{0,1\}^{\mathbb{N}}$ *is uncountable.*

Theorem 17. $\mathbb{R}$ *is uncountable.*