Please solve the following problems and submit your solutions at the *beginning* of lecture on the day that it is due. If the question asks you to "prove" something, you must write a fully-rigorous argument. If you are asked to explain or describe something, you don't need to be as rigorous, but you still must make sure that your ideas are clear.

Please staple this page to the front of your homework, and clearly write your name in the top-right corner.

You may either typeset your solutions using LaTeX, or handwrite them. However, which ever you choose, you must make sure that your solutions are **legible**. Illegible submissions may result in penalties including deducted points, or the requirement that you typeset future assignments. Also note that it is possible for both handwritten and typeset solutions to be illegible.

Some suggestions to increase legibility:

- Use plenty of paper (i.e. don't try to cram everything into one sheet.)
- Use scratch paper for initial work, and only start writing down your final solutions once your ideas have solidified.
- Make sure that your arguments don't get too complicated, organizing your proof into multiple lemmas or subclaims if necessary.

You may collaborate on this assignment, but you must follow the guidelines outlined in the syllabus for writing up your solutions.

This homework is out of 100pts.

(1) [**10 pts**] Let $p(x)$ be a logical formula. Write the following using only the symbols $\forall$, $\exists$, $\wedge$, $\vee$, $\neg$, and $=$ (and parentheses).
  (a) There is exactly one x such that $p(x)$ is true.
  (b) There are at least 2 different choices for x such that $p(x)$ is true.
  (c) There are at most 2 different choices for x such that $p(x)$ is true.

(2) [**10 pts**] Let $p(x, y, z)$, $q(x, y)$ and $r(x, y)$ be logical formulas. Use De Morgan's laws to:
  (a) Rewrite the following proposition so that it does not use any existential quantifiers:

$$\exists x, \forall y, \exists z, p(x, y, z).$$

  Note that it will not be in negational normal form.
  (b) Rewrite the following proposition to put it in negation normal form:

$$\neg\forall x, ((\exists y, p(x, y)) \vee (\exists y, q(x, y)).$$

(3) [**10 pts**] In lecture, we proved that when n is an integer, n^2 always leaves either a 1 or a 2 when divided by 3. We proved by splitting into three cases, one for each remainder when n is divided by 3. However, "Proof Method #2" doesn't directly apply to this case, since it works when there are exactly two cases. Detail a modification of this proof method, called "Proof Method #2b" which works for when you want to split into *three* cases. Like we did with all of the proof methods we outlined in lecture, describe the strategy, outline a template, and justify the method logically.

(4) **[20 pts]** Let $p(x)$ be the logical formula "x is prime" where x ranges over positive integers. Let $q(x)$ be the logical formula "x is irrational" where x ranges over the real numbers. Using p and q, write the following proposition symbolically.

"If n is any prime number, then $\sqrt{n}$ is irrational."

Now prove it, first specifying which proof technique you wish to use, then following the corresponding template outlined in lecture.

(Hint: analyze the proof that $\sqrt{2}$ is irrational)

(5) **[10 pts]** Let $P = \{y \in \mathbb{R} \mid y > 0\}$ be the set of positive real numbers. Prove the following claim:

$$\forall \epsilon \in P, \exists \delta \in P, \forall x \in \mathbb{R}, (|x| < \delta \to |x^2| < \epsilon).$$

(6) **[20pts]** In the following logical formulas, x ranges over the integers $\mathbb{Z}$.

- Let $p(x)$ be "$1 \leq x \leq 3$".
- Let $q(x)$ be "$\exists k \in \mathbb{Z}, (x = 2k)$".
- Let $r(x)$ be "$x^2 = 4$".
- Let $s(x)$ be "$x = 1$".

For each of the following statements, write the logical negation in negation normal form. Then decide which claim (the original or the negation) is true, and prove it.

(a) $\forall x \in \mathbb{Z}, (q(x) \to p(x)$
(b) $\forall x \in \mathbb{Z}, (r(x) \leftrightarrow p(x))$
(c) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, (x \neq y \wedge s(y))$
(d) $\forall x \in \mathbb{Z}, ((p(x) \wedge q(x)) \to r(x))$

(7) **[20pts]** Write what the following statements are saying in natural, idiomatic English. Then determine whether they are true, or false. Prove your answer.

(a) $\forall x \in \mathbb{N}, \exists y \in \mathbb{N}, (2y > x)$
(b) $\exists x \in \mathbb{N}, \forall y \in \mathbb{N}, (x + y = 0)$
(c) $\forall x \in \mathbb{Q}, \forall z \in \mathbb{Q}, \forall y \in \mathbb{Q}, (x < y < z)$
(d) $\forall x \in \mathbb{R}, (x^2 > x)$