Please solve the following problems and submit your solutions at the *beginning* of lecture on Wednesday, May 23. If the question asks you to "prove" something, you must write a fully-rigorous argument. If you are asked to explain or describe something, you don't need to be as rigorous, but you still must make sure that your ideas are clear.

Please staple this page to the front of your homework, and clearly write your name in the top-right corner.

You may either typeset your solutions using LaTeX, or handwrite them. However, which ever you choose, you must make sure that your solutions are **legible**. Illegible submissions may result in penalties including deducted points, or the requirement that you typeset future assignments. Also note that it is possible for both handwritten and typeset solutions to be illegible.

Some suggestions to increase legibility:

- Use plenty of paper (i.e. don't try to cram everything into one sheet.)
- Use scratch paper for initial work, and only start writing down your final solutions once your ideas have solidified.
- Make sure that your arguments don't get too complicated, organizing your proof into multiple lemmas or subclaims if necessary.

You may collaborate on this assignment, but you must follow the guidelines outlined in the syllabus for writing up your solutions.

This homework is out of 100pts.

(1) **[20pts]** For each of the following claims, indicate a proof technique you think is appropriate, and prove the claim using that technique, following the template given in lecture.
   (a) If n is an integer, then $3n^2 + n + 6$ is even.
   (b) If x and y are distinct positive real numbers, then $x + y > 2\sqrt{xy}$.

(2) **[10 pts]** Suppose that I told you that if you win the lottery, you will be rich. What can you conclude in the following four scenarios?
   (a) You win the lottery
   (b) You do not win the lottery
   (c) You will be rich
   (d) You will not be rich

(3) **[10 pts]** Let p and q be propositions.
   (a) Prove that $((p \to q) \wedge p) \to q$ is a tautology.
   (b) Prove that $((p \to q) \vee p) \to q$ is not a tautology.

(4) **[10 pts]** Let p, q, r, and s be propositions. A propositional formula is in *negation normal form* if the only connectives used are $\vee$, $\wedge$ and $\neg$, and the $\neg$ symbols only appear directly in front of propositions. For example, $(\neg p \vee q) \wedge \neg r$ is in negation normal form, but $\neg(p \vee q)$ and $p \to q$ are not.
   (a) Put the following propositional formula in negation normal form:

$$((p \vee \neg q) \to r) \to s$$

(b) Put the negation of the following propositional formula in negation normal form:

$$(p \to q) \land (r \lor \neg q).$$

(5) **[20 pts]** Let's consider propositional formulas constructed from propositions p, q and r. A $\land$-*clause* is a propositional formula which consists p, q, r, or their negations, connected by $\land$. For example, $p \land q \land \neg r$ and $\neg r$ are $\land$-clauses, but $(p \lor q) \land r$ and $p \land \neg(q \land r)$ are not.

A propositional formula is in *conjunctive normal form* (CNF) if it consists of a series of $\land$-clauses connected by $\lor$. For example, $(p \land \neg q) \lor (\neg r \land q) \lor (r \land \neg q)$, $p \lor r$, $p \land q$, and $\neg q$ are all in conjunctive normal form, but $(p \lor q) \land r$ and $\neg(p \lor r)$ are not.

Write the following propositional formulae in CNF, and prove that each one is logically equivalent to the original one. Briefly explain a method you can use to translate a given propositional formula to CNF.

As an example, if we were to write $(p \lor q) \land r$ in CNF, we would get

$$(p \land q \land r) \lor (p \land \neg q \land r) \lor (\neg p \land q \land r).$$

Note that this is not unique. We could change the order of the $\land$-clauses, for example. Also, since $(p \land q \land r) \lor (p \land \neg q \land r)$ is logically equivalent to $p \land r$, we could also write this as

$$(p \land r) \lor (\neg p \land q \land r)$$

and this would also be correct.

(Hint: There are easy ways and hard ways to do this. To discover the easy way, try taking some of the CNF formulae above and fill out their truth tables.)

(a) $(p \land q) \to r$

(b) $(p \to q) \land (r \lor \neg q)$

(6) **[30 pts]** We define the connective $\Box$ with the following truth table:

p	q	$p \Box q$
T	T	F
F	T	T
T	F	T
F	F	T

Explain how every propositional formula involving the propositions $p_1, p_2, ..., p_n$ can be re-written to only use the $\Box$ connective (i.e. no $\neg$, $\lor$, $\land$, $\to$, or $\leftrightarrow$). A fully rigorous proof that your method works would require a tool we have not yet covered (induction). So it will be sufficient to show how you can replace each propositional formula of the form $\neg p$, $p \lor q$, and $p \land q$ with propositional formulas that only use the $\Box$ connective, then write a sentence or two to explain how you would use this to translate the entire propositional formula.

Next, use the method you outline to translate the following propositional formula:

$$\neg(p \lor q)$$

Show that the resulting propositional formula is logically equivalent.