On Randomly Generated Intersecting Hypergraphs

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Abstract

We show that if $r = cn^{1/3}$ and members of $\binom{[n]}{r}$ are chosen sequentially to form an intersecting hypergraph they will, with limiting probability $(1+c^3)^{-1}$, be of maximum size $\binom{n-1}{r-1}$.

1 Introduction

An *intersecting hypergraph* is one in which each pair of edges has a nonempty intersection. Here, we consider r*-uniform hypergraphs* which are those for which all edges contain r vertices.

The motivating idea for this paper is the classical Erdős-Ko-Rado theorem [4] which states that a maximum size r-uniform intersecting hypergraph has $\binom{n-1}{r-1}$ edges if $r \leq n/2$ and $\binom{n}{r}$ edges if $r > n/2$. Furthermore, for $r < n/2$ any maximum-sized family must have the property that all edges contain a common vertex.

In the last four decades this theorem has attracted the attention of many researchers and it has been generalized in many ways. It is worth mentioning for example the famous conjecture of Frankl on the structure of maximum t-intersecting families in a certain range of $n(t, r)$ which was investigated by Frankl and Füredi [6] and completely solved only a few years ago by Ahlswede and Khachatrian [1]. Another type of generalization can be found in [2].

The first attempt (and as far as we know the only one) to 'randomize' this topic was given by Fishburn, Frankl, Freed, Lagarias and Odlyzko [5]. Also note that

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other random hypergraph structures were considered already by Rényi e.g., in [7], he identified the anti-chain threshold. Here we try to continue this line of investigation. Our goal is to describe the structure of random intersecting systems. More precisely, we consider taking edges on-line; that is, one at a time, ensuring that at each stage, the resulting hypergraph remains intersecting. I.e., we consider the following random process: Choose $e_1 \in \binom{[n]}{r}$. Given $\mathcal{F}_i := \{e_1, \ldots, e_i\}$ choose e_{i+1} uniformly at random from among the edges that intersect every edge of $\mathcal{F}_i$. Let $\mathcal{F}$ be the family produced by this procedure.

2 Definitions

Let $[n]$ be the set of vertices of the hypergraph $\mathcal{H}$.

A *ρ-star* is a Δ-system with a one-element kernel; i.e., any pair of hyperedges have the same one-element intersection. A single edge is a 1-star. We say that $\mathcal{H}$ is *fixed* by x if every member of $\mathcal{H}$ contains x.

For any sequence of events $\mathcal{E}_n$, we will say that $\mathcal{E}_n$ occurs with high probability (i.e., **whp**) if $\lim_{n\to\infty} \mathbf{Pr}(\mathcal{E}_n) = 1$.

3 The Erdős-Ko-Rado Threshold

The following theorem determines the threshold for the event that edges chosen online to form an intersecting hypergraph will attain the Erdős-Ko-Rado bound.

Theorem 1. *Let $\mathcal{E}_{n,r}$ be the event that $\left\{|\mathcal{F}| = \binom{n-1}{r-1}\right\} = \{\mathcal{F}$ fixes some element $x \in [n]\}$. For $r < n/2$, this is equivalent to $\mathcal{F}$ fixing some $x \in [n]$. Then if $r = c_n n^{1/3} < n/2$,*

$$\lim_{n\to\infty} \mathbf{Pr}(\mathcal{E}_{n,r}) = \begin{cases} 1 & c_n \to 0 \\ \frac{1}{1+c^3} & c_n \to c \\ 0 & c_n \to \infty \end{cases}.$$

Note: If $r > n/2$, then all of $\binom{[n]}{r}$ is an intersecting hypergraph. If $r = n/2$ then for any $\mathcal{H}$ chosen online to be an intersecting hypergraph, it will have size

$$\binom{n-1}{n/2-1} = \frac{1}{2}\binom{n}{n/2}.$$

In the case of $r = n/2$, however, a vertex will not necessarily be fixed for even $n \geq 4$.

4 Proof of Theorem 1

4.1 Main propositions

Before we prove relevant propositions, we need to define some events.

- Let $\mathcal{A}_i$ be the event that $\mathcal{F}_i$ is an i-star, for $i \geq 1$.
- Let $\mathcal{B}_i$ be the event that $\left\{\cap_{j=1}^{i} e_j \neq \emptyset\right\}$, for $i \geq 3$.
- Let $\mathcal{C}$ be the event that e_3 contains all of $e_1 \cap e_2$ as well as at least one vertex in $(e_1 \setminus e_2) \cup (e_2 \setminus e_1)$.
- Let $\mathcal{D}$ be the event that there is some r-set that intersects all currently chosen edges but fails to contain any vertex in their common intersection.

Proposition 1. *If $r = o(n^{1/2})$ then*

$$\mathbf{Pr}(\mathcal{A}_2) = 1 + o(1).$$

$\diamondsuit$

The fulcrum on which Theorem 1 rests is Proposition 2.

Proposition 2. *If $r = o(n^{1/2})$ then*

$$\mathbf{Pr}(\mathcal{A}_3) = \frac{1 + o(1)}{1 + \frac{(r-1)^3}{n}(1 + o(1))}$$

$\diamondsuit$

Proposition 3. *If $r = o(n^{2/5})$ and $m = O(n^{1/2}/r)$ then*

$$\mathbf{Pr}(\mathcal{A}_m \mid \mathcal{A}_3) = \exp\left\{-\frac{m^2 r^2}{4n} + o(1)\right\}$$

$\diamondsuit$

Remark 1. *Observe that Propositions 1, 2, 3 imply that if $r = d_n n^{1/4}$, then the probability of the event $\mathcal{A}_{r+1}$ approaches $\exp\{-d^4/4\}$ as $d_n \to d$. Furthermore, the occurrence of $\mathcal{A}_{r+1}$ immediately implies $\mathcal{A}_s$ for $s > r + 1$.*

Proposition 4. *If* $r = o(n^{1/2})$ *then*

$$\mathbf{Pr}(\mathcal{C} \mid \mathcal{A}_2) = o(1).$$

$\diamondsuit$

Proposition 5. *If* $r = o(n^{3/8})$ *then*

$$\mathbf{Pr}(\mathcal{B}_{3r} \mid \mathcal{A}_4) = 1 - o(1).$$

$\diamondsuit$

Proposition 6. *If* $r = o(n^{2/5})$ *then*

$$\mathbf{Pr}(\mathcal{D} \mid \mathcal{B}_{3r}, \mathcal{A}_4) = o(1)$$

$\diamondsuit$

Proposition 7. *If* $r = \omega(n^{1/3})$ *(i.e.* $r/n^{1/3} \to \infty$*) and* $r = o(n^{2/3})$ *then*

$$\mathbf{Pr}(\mathcal{B}_3) = o(1)$$

$\diamondsuit$

Proposition 8. *If* $r = \omega(n^{1/2})$ *and* $2\log_2 n \leq m = o(e^{r^2/n})$ *then*

$$\mathbf{Pr}(\mathcal{B}_m) = o(1).$$

$\diamondsuit$

4.2 Using these propositions

Case 1: $r \leq n^{1/3} \log n$.

Suppose first that $c_n \to c$. Then Proposition 1 shows that $\mathcal{A}_2$ occurs **whp**. Given $\mathcal{A}_2$ there are 3 disjoint possibilities

$$\mathcal{A}_3 \,\dot\cup\, \overline{\mathcal{B}_3} \,\dot\cup\, \mathcal{C}. \tag{1}$$

Proposition 4 shows that the conditional probability of $\mathcal{C}$ tends to zero. Proposition 2 shows that $\mathcal{A}_3$ occurs with limiting probability $\frac{1}{1+c^3}$ and so given $\mathcal{A}_2$ the probability of $\overline{\mathcal{B}_3}$ tends to $\frac{c^3}{1+c^3}$. If $\mathcal{B}_3$ does not occur then $\mathcal{F}$ cannot fix an element.

Suppose then that $\mathcal{A}_3$ occurs and $e_1 \cap e_2 \cap e_3 = \{v\}$. We use Proposition 3 with $m = 4$ to show that $\mathcal{A}_4$ occurs with conditional probability $1 - o(1)$. Then, given $\mathcal{A}_4$ we can use Proposition 5 to show that $\mathcal{B}_{3r}$ occurs **whp** and Proposition 6 to show that with conditional probability $1 - o(1)$, $\mathcal{F}$ must fix v.

If $c_n \to 0$ then $\mathcal{A}_3$ occurs **whp** and we conclude as in the previous paragraph that with conditional probability $1 - o(1)$, $\mathcal{F}$ must fix v, where $e_1 \cap e_2 \cap e_3 = \{v\}$.

Now assume that $c_n \to \infty$. We still have $\mathcal{A}_2$ occuring **whp**, but now $\overline{\mathcal{A}_3}$ occurs **whp**. Using decomposition (1) and Proposition 4 to rule out event $\mathcal{C}$ we see that $\overline{\mathcal{B}_3}$ occurs **whp** and so $\mathcal{F}$ cannot fix any element.

Case 2: $n^{1/3} \log n \le r \le n^{1/2} \log n$.
Here we use Proposition 7, which immediately gives that **whp** $\mathcal{F}_3$ has no vertex of degree 3; thus $\mathcal{F}$ cannot fix any element.

Case 3: $n^{1/2} \log n \le r < n/2$.
In this case, we apply Proposition 8 with $m = \exp\left\{\frac{r^2}{3n}\right\}$ and we see that

$$\mathbf{Pr}(\mathcal{B}_m) = O\left(\exp\left\{-\frac{r^2}{3n}\right\}\right) = o(1).$$

So $\mathcal{F}_m$ fails, **whp**, to have a vertex of degree m, in which case $\mathcal{F}$ cannot fix any element. □

5 Proofs of Propositions

5.1 Proof of Proposition 1

First we see that

$$\mathbf{Pr}(\mathcal{A}_1) = 1. \tag{2}$$

$$\begin{aligned} \mathbf{Pr}\left(\mathcal{A}_2 \mid \mathcal{A}_1\right) &= \frac{r\binom{n-r}{r-1}}{\binom{n}{r} - \binom{n-r}{r}} \qquad (3) \\ &= \frac{\frac{rn^{r-1}}{(r-1)!}\left(1 + O\left(\frac{r^2}{n}\right)\right)}{\frac{n^r}{r!}\left(1 - 1 + \frac{r^2}{n} + O\left(\frac{r^3}{n^2}\right)\right)} \\ &= 1 + O\left(\frac{r^2}{n}\right). \end{aligned}$$

□

5.2 Proof of Proposition 2

Continuing as in (3),

$$\mathbf{Pr}\left(\mathcal{A}_{i+1} \mid \mathcal{A}_i\right) = \frac{\binom{n-i(r-1)-1}{r-1}}{\binom{n-1}{r-1} + N_i - i}, \qquad i \geq 2. \tag{4}$$

For $i \geq 2$, the quantity N_i is the number of r sets that intersect all of $\mathcal{F}_i$ but fail to contain the one-vertex kernel of $\mathcal{F}_i$. Thus,

$$(r-1)^i \binom{n-i(r-1)-1}{r-i} \leq N_i \leq (r-1)^i \binom{n-i-1}{r-i}. \tag{5}$$

The lower bound comes from taking a single vertex (not the kernel) from each of the edges and $r-i$ vertices from the remainder of the vertex set. The upper bound comes from taking one vertex (not the kernel) from each of the edges and $r-i$ other non-kernel vertices.

Simple computations give, for $r = o(n^{1/2})$,

$$N_2 \;=\; (1+o(1))\frac{(r-1)^3}{n}\binom{n-1}{r-1}. \tag{6}$$

$$N_3 \;\leq\; (1+o(1))\binom{n-1}{r-1}. \tag{7}$$

$$\binom{n-i(r-1)-1}{r-1} \;=\; (1+o(1))\binom{n-1}{r-1}. \tag{8}$$

It follows from (4), (6), (7) and (8) that

$$\mathbf{Pr}(\mathcal{A}_3 \mid \mathcal{A}_2) = \frac{1+o(1)}{1+\frac{(r-1)^3}{n}(1+o(1))}.$$

Proposition 1 then gives that

$$\mathbf{Pr}(\mathcal{A}_3) = \frac{1+o(1)}{1+\frac{(r-1)^3}{n}(1+o(1))}. \tag{9}$$

$\square$

5.3 Proof of Proposition 3

We estimate for $3 \leq i \leq r$:

$$\frac{(r-1)^i\binom{n-i-1}{r-i}}{\binom{n-1}{r-1}} \leq \frac{r^i\binom{n-1}{r-i}}{\binom{n-1}{r-1}} = O\left(\frac{r^{2i-1}}{n^{i-1}}\right). \tag{10}$$

It then follows from (4), (5) and (10) that for $3 \le i \le r$,

$$\begin{aligned}
\mathbf{Pr}(\mathcal{A}_{i+1} \mid \mathcal{A}_i) &= \frac{\binom{n-i(r-1)-1}{r-1}}{\binom{n-1}{r-1}\left(1+O\left(\frac{r^{2i-1}}{n^{i-1}}\right)\right)} \\
&= 1 - \frac{ir^2}{2n} + O\left(\frac{i^2r^3}{n^2} + \frac{r^{2i-1}}{n^{i-1}}\right). \qquad (11)
\end{aligned}$$

Equation (11) implies that

$$\begin{aligned}
\mathbf{Pr}(\mathcal{A}_{m+1} \mid \mathcal{A}_3) &= \prod_{i=3}^{m} \mathbf{Pr}(\mathcal{A}_{i+1} \mid \mathcal{A}_i) \\
&= \prod_{i=3}^{m} \left(1 - \frac{ir^2}{2n} + O\left(\frac{i^2r^3}{n^2} + \frac{r^{2i-1}}{n^{i-1}}\right)\right) \\
&= \prod_{i=3}^{m} \exp\left\{-\frac{ir^2}{2n} + O\left(\frac{i^2r^4}{n^2} + \frac{r^{2i-1}}{n^{i-1}}\right)\right\} \\
&= \exp\left\{-\frac{m^2r^2}{4n} + o(1)\right\}.
\end{aligned}$$

□

5.4 Proof of Proposition 4

A simple computation suffices:

$$\mathbf{Pr}(\mathcal{C} \mid \mathcal{A}_2) \le \frac{2r\binom{n-2}{r-2}}{\binom{n-1}{r-1} - 2} \le \frac{2r^2}{n - 2r\binom{n-1}{r-1}^{-1}} = O\left(\frac{r^2}{n}\right).$$

□

5.5 Proof of Proposition 5

Assuming that both $\mathcal{A}_4$ and $\mathcal{B}_i$ occur for $i \ge 4$, there are at most $(r-1)^4\binom{n-1}{r-4}$ r-sets which do not contain v and which meet e_1, e_2, e_3, e_4. On the other hand there are $\binom{n-1}{r-1} - i$ r-sets which contain v and are not edges of $\mathcal{F}_i$. As a result, for $i \ge 4$,

$$\mathbf{Pr}(\overline{\mathcal{B}_{i+1}} \mid \mathcal{B}_i, \mathcal{A}_4) \le \frac{(r-1)^4\binom{n-1}{r-4}}{\binom{n-1}{r-1} - i} \le \frac{2r^7}{n^3}. \qquad (12)$$

Thus

$$\begin{aligned}
\mathbf{Pr}(\mathcal{B}_{3r} \mid \mathcal{A}_4) &= \prod_{i=4}^{3r-1} \mathbf{Pr}(\mathcal{B}_{i+1} \mid \mathcal{B}_i, \mathcal{A}_4) \\
&\geq \prod_{i=4}^{3r-1} \left(1 - \frac{2r^7}{n^3}\right) \\
&\geq 1 - \frac{6r^8}{n^3}.
\end{aligned}$$

□

5.6 Proof of Proposition 6

Assume that $\mathcal{B}_{3r} \cap \mathcal{A}_4$ occurs and that v is the unique vertex of degree $3r$ in $\mathcal{F}_{3r}$. We show that **whp** $v \in e_i$ for $i > 3r$.

Claim 1. *Suppose that $\mathcal{B}_{3r} \cap \mathcal{A}_4$ occurs. Then $e_i' = e_i \setminus \{v\}, 1 \leq i \leq 3r$ is a collection of $3r$ randomly chosen $(r-1)$-sets from $[n] \setminus \{v\}$.*

The claim can be argued as follows: e_i is chosen uniformly from all r-sets which meet $e_1, e_2, \ldots, e_{i-1}$. If we add the condition $v \in e_i$ i.e. $\mathcal{B}_i$ occurs, then e_i is equally likely to be any such r-set containing v. □

Recall that $\mathcal{D}$ is the event that there is an r-set which meets all edges but does not contain the kernel. Then

$$\begin{aligned}
\mathbf{Pr}(\mathcal{D} \mid \mathcal{B}_{3r}, \mathcal{A}_4) &\leq \binom{n-1}{r} \left(1 - \frac{\binom{n-r-1}{r-1}}{\binom{n-1}{r-1}}\right)^{3r} \\
&\leq \left(\frac{ne}{r}\right)^r \left(\frac{r^2}{n-2r}\right)^{3r} \\
&= \left(\frac{ner^5}{(n-2r)^3}\right)^r \\
&\leq \left(\frac{2er^5}{n^2}\right)^r
\end{aligned}$$

□

5.7 Proof of Proposition 7

We show that $\mathbf{Pr}(\mathcal{B}_3) = o(1)$. We write

$$\mathbf{Pr}(\mathcal{B}_3) = \sum_{i=1}^{r-1} f(i)g(i) \tag{13}$$

where

$$\begin{aligned} f(i) &= \mathbf{Pr}(|e_1 \cap e_2| = i) \\ &= \frac{\binom{r}{i}\binom{n-r}{r-i}}{\binom{n}{r} - \binom{n-r}{r}} \end{aligned} \tag{14}$$

and

$$\begin{aligned} g(i) &= \mathbf{Pr}(\mathcal{B}_3 \mid |e_1 \cap e_2| = i) \\ &= \frac{\binom{n}{r} - \binom{n-i}{r}}{\binom{n}{r} - 2\binom{n-r}{r} + \binom{n-2r+i}{r}} \end{aligned} \tag{15}$$

Now for $0 \leq s \leq 2r$ we have

$$\begin{aligned} \frac{\binom{n-s}{r}}{\binom{n}{r}} &= \prod_{j=0}^{r-1} \left(1 - \frac{s}{n-j}\right) \\ &= \prod_{j=0}^{r-1} \exp\left\{-\frac{s}{n} + O\left(\frac{r^2}{n^2}\right)\right\} \\ &= \exp\left\{-\frac{rs}{n} + O\left(\frac{r^3}{n^2}\right)\right\}. \end{aligned} \tag{16}$$

Furthermore,

$$\begin{aligned} \frac{\binom{r}{i}\binom{n-r}{r-i}}{\binom{n}{r}} &\leq \frac{r^i}{i!} \cdot \frac{\binom{n-r}{r-i}}{\binom{n-r}{r}} \cdot \frac{\binom{n-r}{r}}{\binom{n}{r}} \\ &\leq \frac{r^i}{i!} \cdot \frac{r^i}{(n-2r)^i} \exp\left\{-\frac{r^2}{n} + O\left(\frac{r^3}{n^2}\right)\right\}. \end{aligned}$$

Thus

$$f(i) \leq \frac{r^{2i}}{i!(n-2r)^i} \cdot \frac{1+o(1)}{\exp\left\{\frac{r^2}{n}\right\} - 1}. \tag{17}$$

Using (16) in (15) we see that

$$\begin{aligned} g(i) &= \frac{1 - \exp\left\{-\frac{ir}{n} + O\left(\frac{r^3}{n^2}\right)\right\}}{1 - 2\exp\left\{-\frac{r^2}{n} + O\left(\frac{r^3}{n^2}\right)\right\} + \exp\left\{-\frac{r(2r-i)}{n} + O\left(\frac{r^3}{n^2}\right)\right\}} \\ &\leq (1+o(1)) \frac{\frac{ir}{n} + O\left(\frac{r^3}{n^2}\right)}{\left(1 - \exp\left\{-\frac{r^2}{n}\right\}\right)^2}. \end{aligned}$$

So,

$$
\begin{aligned}
\sum_{i=1}^{r-1} f(i)g(i) &\leq (1+o(1))\frac{\exp\left\{\frac{2r^2}{n}\right\}}{\left(\exp\left\{\frac{r^2}{n}\right\}-1\right)^3}\sum_{i=1}^{r-1}\frac{r^{2i}}{i!(n-2r)^i}\left(\frac{ir}{n}+O\left(\frac{r^3}{n^2}\right)\right) \\
&= O\left(\frac{\exp\left\{\frac{2r^2}{n}\right\}}{\left(\exp\left\{\frac{r^2}{n}\right\}-1\right)^3}\frac{r^3}{n^2}\exp\left\{\frac{r^2}{n-2r}\right\}\right) \\
&= O\left(\frac{r^3}{n^2}\cdot\frac{1}{\left(1-\exp\left\{-\frac{r^2}{n}\right\}\right)^3}\right) \\
&= o(1).
\end{aligned}
$$

□

5.8 Proof of Proposition 8

Consider m members of $\binom{[n]}{r}$ being chosen at random (without replacement).

The probability that these m edges fail to form an intersecting family is at most

$$
\binom{m}{2}\frac{\binom{n-r}{r}}{\binom{n}{r}} \leq \frac{m^2}{2}\left(1-\frac{r}{n}\right)^r \leq \frac{m^2}{2}\exp\left\{-\frac{r^2}{n}\right\}
$$

Let us take

$$
m = \exp\left\{\frac{r^2}{3n}\right\}.
$$

For $r = \omega(\sqrt{n})$ we can think of $\mathcal{F}_m$ having the same distribution as m randomly chosen r-sets, conditional on the event of probability $1-o(1)$ that $\mathcal{F}_m$ is intersecting.

Using $r < n/2$, the probability that $\mathcal{F}_m$ has a vertex of degree m is at most

$$
\begin{aligned}
\frac{1}{2}\exp\left\{-\frac{r^2}{3n}\right\} + n\left(\frac{\binom{n-1}{r-1}}{\binom{n}{r}}\right)^m &= O\left(\exp\left\{-\frac{r^2}{3n}\right\}\right) + r^m n^{1-m} \\
&= O\left(\exp\left\{-\frac{r^2}{3n}\right\}\right) + n2^{-m}.
\end{aligned}
$$

□

6 Open Problem

It is known that a maximal intersecting system, i.e, a system to which we can not add any additional edge without making it non-intersecting, may have various structures. Thus we finish by posing the following problem.

Problem: What is the structure of $\mathcal{F}$ in different ranges of $n^{1/3} \ll r < n/2$?

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